VII

Adjoint Situations

The slogan is "Adjoint functors arise everywhere".

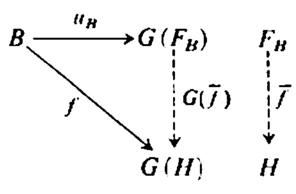
——S. Mac Lane†

In Chapter I we claimed that category theory allows one to make precise the notion of "universality". In this chapter—the most important of the text—we will show why this is so. Here we investigate adjoint situations—situations occurring so frequently and in so many diverse areas of mathematics that they are regarded as perhaps the most useful of all categorical notions.

§26 UNIVERSAL MAPS

The following well-known example motivates our definition of universal maps:

Let G: **Grp** → **Set** be the forgetful (i.e., "grounding") functor, and let B be a set. Then there exists a group F_B (called the "free group on B") and a function $u_B: B \to G(F_B)$ such that for any group H and any function $f: B \to G(H)$, there exists a unique group homomorphism $\bar{f}: F_B \to H$ such that the triangle

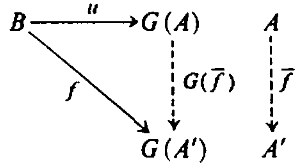


commutes.

† From *Categories for the Working Mathematician.*

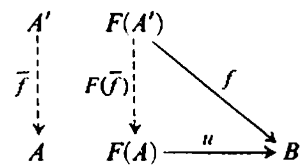
26.1 DEFINITION

Let $G: \mathscr{A} \to \mathscr{B}$ be a functor and let $B \in Ob(\mathscr{B})$. A pair (u, A) with $A \in Ob(\mathscr{A})$ and $u: B \to G(A)$ is called a **universal map for** B **with respect to** G (or a G-**universal map for** B) provided that for each $A' \in Ob(\mathscr{A})$ and each $f: B \to G(A')$ there exists a unique $\mathscr{A}$-morphism $\bar{f}: A \to A'$ such that the triangle



commutes.

DUALLY: If $F: \mathscr{A} \to \mathscr{B}$ is a functor and $B \in Ob(\mathscr{B})$, then a pair (A, u) is called a **co-universal map for** B **with respect to** F (or an F-**(co-universal) map for** B) provided that (u, A) is a universal map for B with respect to $F^{op}: \mathscr{A}^{op} \to \mathscr{B}^{op}$; i.e., provided that $u: F(A) \to B$ and for each $\mathscr{A}$-object A' and each $\mathscr{B}$-morphism $f: F(A') \to B$, there exists a unique morphism $\bar{f}: A' \to A$ such that the triangle



commutes.

26.2 EXAMPLES

Motivated by Theorems A, B, C, and D of Chapter I, we give three major varieties of examples of universal maps:

(1) *Universal maps for forgetful functors; free objects*

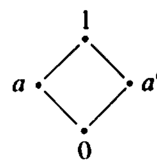
(a) *Forgetful functors to* **Set**

Let $(\mathscr{A}, U)$ be any of the concrete categories in the following table. Then for each set B there exists a U-universal map (u_B, A_B). In the table we list

$(\mathscr{A}, U)$	A_B
Set	the set B
SGrp	the free semigroup generated by B
Grp	the free group generated by B
R-**Mod**	the free R-module generated by B
R-**Alg**	the free R-algebra generated by B
commutative rings	the polynomial ring $\mathbf{Z}[B]$ over $\mathbf{Z}$ with set of indeterminants B
Top	the discrete space with underlying set B
POS	the "trivial" partially ordered set on B (i.e., the order is equality)
CompT$_2$	the Stone-Čech compactification of the discrete space with underlying set B

only A_B, since in each case $u_B: B \to U(A_B)$ is the obvious inclusion of B into the underlying set of A_B. The objects A_B are usually called "free objects" (more precisely "U-free objects") and the functions $u_B: B \to U(A_B)$ are often called "insertions of the generators".

Of special interest are the "free objects with one generator", i.e., the universal maps (u_B, A_B) for a singleton set $B = \{a\}$. In this case u_B just labels one distinguished element ("the generator") in the underlying set of A_B.

$(\mathscr{A}, U)$	Generator	U-free object with this singleton "generator"
SGrp	1	$(\mathbf{N} - \{0\}, +)$
Mon	1	$(\mathbf{N}, +)$
Grp	1	$(\mathbf{Z}, +)$
Ab	1	$(\mathbf{Z}, +)$
Rng	X	$\mathbf{Z}[X]$
commutative rings	X	$\mathbf{Z}[X]$
BooAlg	a	

(b) *Other forgetful functors*
If G: **Rng** $\to$ **Mon** is the functor which "forgets addition" and sends each ring $(R, +, \cdot, 1)$ to its multiplicative monoid $(R, \cdot, 1)$, then for each monoid B there is a G-universal map (u_B, A_B), where A_B is the monoid ring $\mathbf{Z}[B]$ of B over the additive group $(\mathbf{Z}, +)$, and u_B is the obvious embedding. Likewise if H: **Rng** $\to$ **Ab** is the functor that "forgets multiplication" and sends each ring $(R, +, \cdot, 1)$ to the abelian group $(R, +)$, then for each abelian group B there exists an H-universal map (u_B, A_B) where A_B is the tensor ring over B and u_B is the obvious embedding.

(2) *Universal maps for inclusion functors*

If $\mathscr{A}$ is a subcategory of $\mathscr{B}$, $E: \mathscr{A} \to \mathscr{B}$ is the inclusion functor, and B is a $\mathscr{B}$-object, then (u_B, A_B) is called an **$\mathscr{A}$-reflection of B** provided that it is an E-universal map for B. Reflections are so abundant that they deserve special attention (see Chapter X). In each of the examples listed below, $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ such that each $\mathscr{B}$-object B has an $\mathscr{A}$-reflection (u_B, A_B). We list only A_B since the morphism $u_B: B \to A_B$ is always obvious. The examples are arranged in several groups according to the type of construction involved for forming A_B.

(a) "*Completions*"

$\mathscr{A}$	$\mathscr{B}$	A_B
complete metric spaces	metric spaces (and uniformly continuous maps)	the metric completion of B
complete uniform spaces	uniform spaces (and uniformly continuous maps)	the uniform completion of B
CompT$_2$	**CRegT$_2$**	the Stone-Čech compactification, βB, of B
realcompact Hausdorff spaces	**CRegT$_2$**	the Hewitt realcompactification, υB, of B
zero-dimensional compact Hausdorff spaces	zero-dimensional Hausdorff spaces	the Banaschewski zero-dimensional compactification, ζB, of B
BanSp$_1$	**NLinSp**	the completion of the normed linear space B
Ab	cancellative abelian semigroups	the group of quotients of B
Field	integral domains and injective ring homomorphisms	the field of quotients of B
complete partially-ordered sets	partially-ordered sets (and sup-preserving functions)	the order-completion of B†

† The order-completion of B is the subset of the partially-ordered set $(\mathscr{P}(B), \subset)$ consisting of all those subsets A of B which satisfy the following two conditions:

(1) $x \in B$, $a \in A$, and $x \leq a \Rightarrow x \in A$

(2) $X \subset A \Rightarrow sup_B X \in A$.

This completion is in general different from the well-known MacNeille completion.

(b) *Quotients formed by "factoring out" subobjects*

$\mathscr{A}$	$\mathscr{B}$	A_B
Ab	**Grp**	B/B' (where B' is the commutator subgroup of B)
torsion-free abelian groups	**Ab**	$B/\hat{B}$ (where $\hat{B}$ is the torsion subgroup of B)
$\mathbf{Ab}^{(n)}$ (= abelian groups G with $nG = 0$)	**Ab**	B/nB
reduced rings (no nilpotent elements except 0)	commutative rings	$B/r(B)$ (where $r(B) = \{x \mid x \in B$ such that there is some $n \in \mathbb{N}$ with $x^n = 0\}$ is the nilradical of B)

(c) "*Identifications*"

$\mathscr{A}$	$\mathscr{B}$	A_B
POS	quasi-ordered sets (and monotone functions)	$B/\sim$ (where $\sim$ is the equivalence relation on B defined by $b \sim b'$ if and only if $b \leq b'$ and $b' \leq b$)
T_0 topological spaces	**Top**	$B/\sim$ (where $\sim$ is the equivalence relation on B defined by $b \sim b'$ if and only if $b \in \{b'\}^-$ and $b' \in \{b\}^-$)
metric spaces	pseudo-metric spaces	$B/\sim$ (where $\sim$ is the equivalence relation on B defined by $b \sim b'$ if and only if $d(b, b') = 0$

(d) "*Modifications*" *of structure on underlying sets*

$\mathscr{A}$	$\mathscr{B}$	A_B
regular topological spaces	**Top**	the regular modification of B
locally convex linear topological spaces	**LinTop**	the locally convex modification of B

(3) *Universal maps for internal Hom-functors*

For the categories $\mathscr{A}$ listed below, let $Hom(C, \_\_): \mathscr{A} \to \mathscr{A}$ be the internal *Hom*-functor with respect to the $\mathscr{A}$-object C, and let $B \in Ob(\mathscr{A})$. Then (u_B, A_B) is a universal map for B with respect to $Hom(C, \_\_)$, where A_B and u_B are as defined in the table that follows.

$\mathscr{A}$	A_B	$u_B: B \to Hom(C, A_B)$
Set	$B \times C$	$(u_B(b))(c) = (b, c)$
R-**Mod** (where R is commutative)	$B \otimes_R C$	$(u_B(b))(c) = b \otimes c$
Top (where C is locally compact Hausdorff)	$B \times C$	$(u_B(b))(c) = (b, c)$
pTop (where C is locally compact Hausdorff)†	$B \wedge C$ (smash product)	$(u_B(b))(c) = \begin{cases} \text{the distinguished point, if } (b, c) \text{ is in the wedge of } B \text{ and } C; \\ (b, c), \text{ otherwise} \end{cases}$
Cat or $\mathscr{CAT}$	$B \times C$	where $(u_B(b))(c) = (b, c)$, for each $b \in Ob(B)$; and if $m: b \to b'$, $u_B(m): u_B(b) \to u_B(b')$ is the natural transformation defined by: $u_B(m)_C = (m, 1_C)$
locally convex normed linear spaces	$B \otimes C$	where $(u_B(b))(c) = b \otimes c$

† If, for this example, one chooses the circle S^1 for C, then $Hom(C, B)$ is called the **loop space** of B and $B \wedge C$ is called the **suspension of** B.

(4) *Co-universal maps*

In each of the following examples $E: \mathscr{A} \hookrightarrow \mathscr{B}$ is the inclusion functor, B is a $\mathscr{B}$-object, and (A_B, u_B) is an E-(co-universal) map for B (where $u_B: E(A_B) \to B$ is the "obvious" morphism).

$\mathscr{A}$	$\mathscr{B}$	A_B
abelian torsion groups	Ab	the torsion subgroup of B
$\mathbf{Ab}^{(n)}$ (see (2)b)	Ab	the subgroup of B composed of all elements x of B with $nx = 0$
nilrings	commutative rings	the nilradical of B
locally connected spaces	Top	the "locally connected refinement" of B
compactly generated spaces	Top	the "compactly generated refinement" of B
finitely-generated spaces	Top	the topological space whose underlying set is that of B and whose open sets are the intersections of open sets in B
trivially partially-ordered sets	POS	the trivial partially-ordered set with the same underlying set as B

It is clear that the preceding examples (1)–(4) correspond to many diverse and important classical constructions in mathematics. Moreover, it can be shown that colimits (resp. limits) of functors $D: I \to \mathscr{A}$ can be interpreted as G-universal maps (resp. G-(co-universal maps)) relative to a suitable functor $G: \mathscr{A} \to \mathscr{A}^I$ (see Exercise 26I). The fact that all of these constructions can be described in essentially the same way once again underscores the significance and usefulness of categorical language. It also clearly classifies the notion of universal maps (and as we shall see later, adjoint situations) as an extremely important mathematical concept. Later (§28) we shall also see that most of the above constructions can actually be *accomplished* in essentially the same way; i.e., that there are very general theorems that yield most of the above examples as special cases.

Next we introduce the concept of "generation", which is a generalization of some earlier notions (see examples 26.4) and which will be investigated more thoroughly in later sections.

26.3 DEFINITION

Let $G: \mathcal{A} \to \mathcal{B}$ be a functor.

(1) A morphism $g: B \to G(A)$ is said to G-**generate** A provided that whenever $A \overset{r}{\underset{s}{\rightrightarrows}} A'$ are $\mathcal{A}$-morphisms such that $G(r) \circ g = G(s) \circ g$, then $r = s$.

(2) A morphism $g: B \to G(A)$ is said to **extremally** G-**generate** A provided that

(i) it G-generates A, and

(ii) (*Extremal condition*): whenever there is an $\mathcal{A}$-monomorphism m with codomain A and a $\mathcal{B}$-morphism f such that $g = G(m) \circ f$, then m must be an isomorphism.

(3) A $\mathcal{B}$-object B is said to G-**generate** (resp. **extremally** G-**generate**) the $\mathcal{A}$-object A provided that there is some $\mathcal{B}$-morphism $g: B \to G(A)$ that G-generates (resp. extremally G-generates) A.

26.4 EXAMPLES

(1) If $G: \mathcal{A} \to \mathcal{A}$ is the identity functor, then $g: B \to A$ G-generates A if and only if g is an epimorphism; g extremally G-generates A if and only if g is an extremal epimorphism.

(2) If I is a discrete category and $G: \mathcal{A} \to \mathcal{A}^I$ is the "constant functor" functor (15.8), then $(f_i: A_i \to A)_I$ G-generates A if and only if (f_i, A) is an epi-sink in $\mathcal{A}$; and $(f_i)_I$ extremally G-generates A if and only if (f_i, A) is an (extremal epi)-sink in $\mathcal{A}$ (19.1).

(3) Let $(\mathcal{A}, U)$ be any one of the concrete categories **Grp**, R-**Mod**, **Lat**, **BooAlg**, **SGrp**, **Mon**, or **Rng**. If $g: B \to U(A)$ is a function from a set B into the underlying set $U(A)$ of an $\mathcal{A}$-object A, then g extremally generates A if and only if $g[B]$ "generates A" in the algebraic sense; i.e., provided that A does not contain a proper subobject whose underlying set contains $g[B]$. In the case of groups and R-modules, whenever $g: B \to U(A)$ generates A, it also extremally generates A. This is not the case for semigroups, monoids, and rings. In these cases, $g: B \to U(A)$ generates A if and only if the embedding $e: C \to A$ of the subobject C of A, generated (in the algebraic sense) by $g[B]$, is an epimorphism. [These examples motivate our use of the term "generates".]

(4) Let $(\mathcal{A}, U)$ be the concrete category **Top** (resp. **Top$_2$**). If $g: B \to U(A)$ is a function from a set B into the underlying set of a space (resp. Hausdorff space) A, then g generates A if and only if it is surjective (resp. $g[B]$ is dense in A); and it extremally generates A if and only if it is surjective and A is a discrete space.

26.5 PROPOSITION

If $\mathcal{A}$ has equalizers and $G: \mathcal{A} \to \mathcal{B}$ preserves them, then a $\mathcal{B}$-morphism $g: B \to G(A)$ extremally G-generates A if it satisfies the extremal condition (ii) *of Definition* 26.3(2). (Cf. 17.14 and 19.4.)

Proof: We need only show that g G-generates A. Suppose that $A \overset{r}{\underset{s}{\rightrightarrows}} A'$

are $\mathscr{A}$-morphisms such that $G(r) \circ g = G(s) \circ g$. Let $(K, k) \approx Equ(r, s)$. Then k is a monomorphism and since G preserves equalizers,

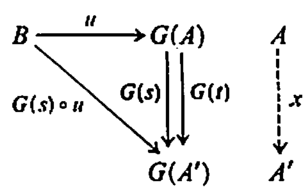
$$(G(K), G(k)) \approx Equ(G(r), G(s)).$$

Thus by the definition of equalizer, there is a morphism $h\colon B \to G(K)$ such that $g = G(k) \circ h$. Hence by the extremal condition k is an isomorphism, so that $r = s$. □

26.6 PROPOSITION

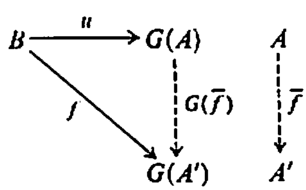
If (u, A) is a G-universal map for B, then u extremally G-generates A.

Proof: If s and t are $\mathscr{A}$-morphisms such that $G(s) \circ u = G(t) \circ u$, then by the definition of universal maps there is a unique morphism x such that $G(x) \circ u$ is $G(s) \circ u$. Hence $x = s = t$.



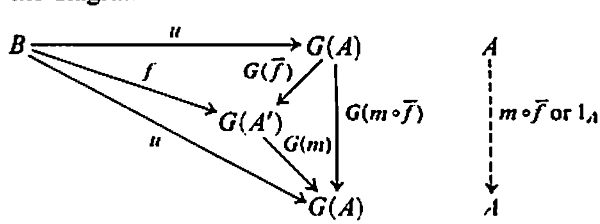
Thus u G-generates A.

To show the extremal condition, suppose that there is a $\mathscr{B}$-morphism f and an $\mathscr{A}$-monomorphism m with codomain A such that $u = G(m) \circ f$. By the definition of universality there is a morphism $\bar{f}$ such that the diagram



commutes.

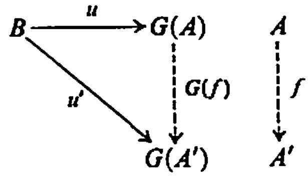
Hence the diagram



also commutes. Consequently by the uniqueness condition in the definition of universal maps $m \circ \bar{f} = 1_A$. Thus m is a retraction and a monomorphism; i.e., an isomorphism. □

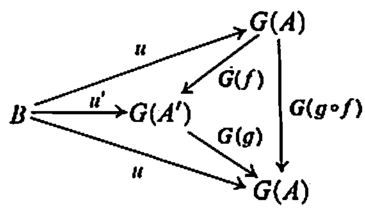
26.7 PROPOSITION

Universal maps are essentially unique; i.e., if $G: \mathscr{A} \to \mathscr{B}$ and each of (u, A) and (u', A') is a G-universal map for some $B \in Ob(\mathscr{B})$, then there is a unique isomorphism $f: A \to A'$ such that the triangle

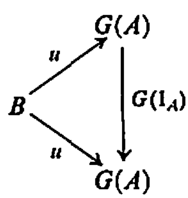


commutes.

Proof: By the definition of universal map there are unique morphisms $f: A \to A'$ and $g: A' \to A$ such that the diagram



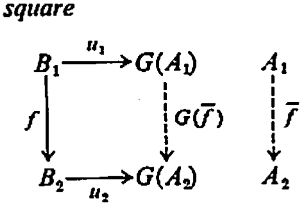
commutes. But so does the triangle



Thus, since u G-generates A (26.6), it follows that $1_A = g \circ f$. Similarly one can show that $1_{A'} = f \circ g$. Consequently f is an isomorphism. □

26.8 LEMMA

If $G: \mathscr{A} \to \mathscr{B}$, $B_1, B_2 \in Ob(\mathscr{B})$, and (u_i, A_i) is a G-universal map for B_i, $i = 1, 2$; then for each morphism $f: B_1 \to B_2$ there exists a unique morphism $\bar{f}: A_1 \to A_2$ such that the square



commutes. □

If a functor $G: \mathscr{A} \to \mathscr{B}$ has the property that each $\mathscr{B}$-object has a G-universal map, then the preceding lemma will enable us to define a functor

$F: \mathscr{B} \to \mathscr{A}$ (see Theorem 26.11). The next somewhat technical lemma will enable us (among other things) to prove that F preserves colimits.

26.9 LEMMA

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad D \quad} & \mathscr{B} \\ & F \searrow \quad \Big\downarrow u=(u_C) \quad \nearrow G & \\ & \mathscr{A} & \end{array}$$

Let $D: \mathscr{C} \to \mathscr{B}$ and $G: \mathscr{A} \to \mathscr{B}$ be functors, and for each $C \in Ob(\mathscr{C})$ let (u_C, A_C) be a G-universal map for $D(C)$.

(1) *Then there exists a unique functor $F: \mathscr{C} \to \mathscr{A}$ such that*
   (i) *for each $C \in Ob(\mathscr{C})$, $F(C) = A_C$; and*
   (ii) *$u = (u_C)$ is a natural transformation from D to $G \circ F$.*

(2) *Let $((k_C), K)$ be a colimit of D.*
   (i) *If $((k'_C), K')$ is a colimit of F, then there exists a unique $\mathscr{B}$-morphism $u': K \to G(K')$ such that for each $C \in Ob(\mathscr{C})$ the square*

$$\begin{array}{ccccc} D(C) & \xrightarrow{u_C} & G \circ F(C) & & F(C) \\ {\scriptstyle k_C}\downarrow & & \downarrow{\scriptstyle G(k'_C)} & & \downarrow{\scriptstyle k'_C} \\ K & \xrightarrow[u']{} & G(K') & & K' \end{array}$$

commutes. In addition, (u', K') is a G-universal map for K.
   (ii) *Conversely, if (u', K') is a G-universal map for K, then for each $C \in Ob(\mathscr{C})$, there exists a unique $k'_C: F(C) \to K'$ such that the above square commutes. In addition $((k'_C), K')$ will be a colimit of F.*

Proof:

(*1*). If $C \xrightarrow{f} C'$, then $D(C) \xrightarrow{D(f)} D(C')$, so that by the above lemma (26.8) there is a unique morphism $\bar{f}: A_C \to A_{C'}$ such that the square

$$\begin{array}{ccccc} D(C) & \xrightarrow{u_C} & G(A_C) & & A_C \\ {\scriptstyle D(f)}\downarrow & & \downarrow{\scriptstyle G(\bar{f})} & & \downarrow{\scriptstyle \bar{f}} \\ D(C') & \xrightarrow[u_{C'}]{} & G(A_{C'}) & & A_{C'} \end{array}$$

commutes.

Define $F(f) = \bar{f}$. Clearly, by the uniqueness, $F: Mor(\mathscr{C}) \to Mor(\mathscr{A})$ is a function, $F(1_C) = 1_{A_C}$, and if F is a functor it is the unique one for which $F(C) = A_C$ and each square above commutes. Since the squares do commute, $u = (u_C): D \to G \circ F$ will be a natural transformation. Thus it remains to be shown that F preserves compositions. Suppose that $C \xrightarrow{f} C' \xrightarrow{g} C''$. Then since G and D preserve compositions, each of $\bar{g} \circ \bar{f}$ and $\overline{g \circ f}$ is a morphism x such that the square

$$\begin{array}{ccccc} D(C) & \xrightarrow{u_C} & G(A_C) & \qquad & A_C \\ {\scriptstyle D(g \circ f)}\downarrow & & \downarrow{\scriptstyle G(x)} & & \downarrow{\scriptstyle x} \\ D(C'') & \xrightarrow[u_{C''}]{} & G(A_{C''}) & & A_{C''} \end{array}$$

commutes.

Thus since u_C G-generates A_C (26.6), $\bar{g} \circ \bar{f} = \overline{g \circ f}$; i.e., $F(g) \circ F(f) = F(g \circ f)$.

(2). (i) By the above definition of F, and the fact that G preserves compositions, for each $g\colon C \to C'$ we have commutativity of the diagram

$$\begin{array}{ccccc} D(C) & \xrightarrow{u_C} & G(F(C)) & \xrightarrow{G(k'_C)} & \\ {\scriptstyle D(g)}\downarrow & & \downarrow{\scriptstyle G(F(g))} & & G(K') \\ D(C') & \xrightarrow[u_{C'}]{} & G(F(C')) & \xrightarrow{G(k'_{C'})} & \end{array}$$

Thus $((G(k'_C) \circ u_C), G(K'))$ is a natural sink for D, so that there is a unique morphism $u'\colon K \to G(K')$ such that for each $C \in Ob(\mathscr{C})$, the square

$$\begin{array}{ccc} D(C) & \xrightarrow{u_C} & G(F(C)) \\ {\scriptstyle k_C}\downarrow & & \downarrow{\scriptstyle G(k'_C)} \\ K & \xrightarrow[u']{} & G(K') \end{array}$$

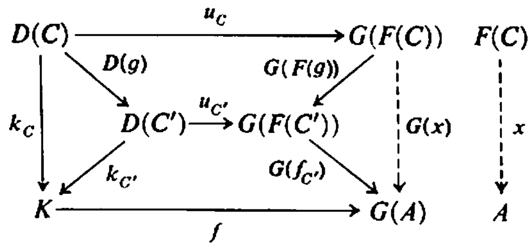
commutes.

To show that (u', K') is a G-universal map for K, suppose that $f\colon K \to G(A)$. Then since the $(u_C, F(C))$ are universal maps, for each $C \in Ob(\mathscr{C})$ there is a unique $f_C\colon F(C) \to A$ such that the triangle

$$\begin{array}{ccccc} D(C) & \xrightarrow{u_C} & G(F(C)) & \qquad & F(C) \\ & {\scriptstyle f \circ k_C}\searrow & \downarrow{\scriptstyle G(f_C)} & & \downarrow{\scriptstyle f_C} \\ & & G(A) & & A \end{array}$$

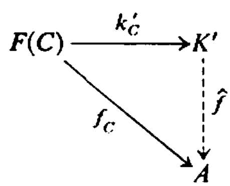
commutes.

Now if $g\colon C \to C'$, then each of f_C and $f_{C'} \circ F(g)$ is a morphism $x\colon F(C) \to A$ for which the outer square of the diagram



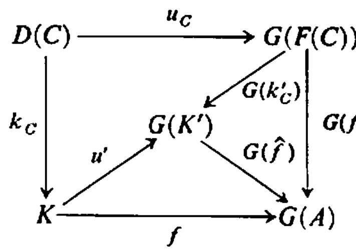
commutes.

Thus, by uniqueness, $f_C = f_{C'} \circ F(g)$, so that $((f_C), A)$ is a natural sink for F. Since $((k'_C), K')$ is a colimit of F, there is a unique $\hat{f}\colon K' \to A$ such that for each $C \in Ob(\mathscr{C})$, the triangle



commutes.

Consequently, for the diagram

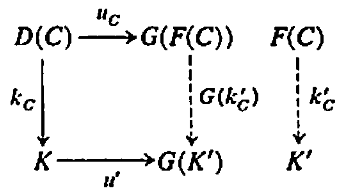


we have commutativity of the outer square, the upper quadrilateral, and the right-hand triangle; from which it follows that for each $C \in Ob(\mathscr{C})$,

$$f \circ k_C = G(\hat{f}) \circ u' \circ k_C.$$

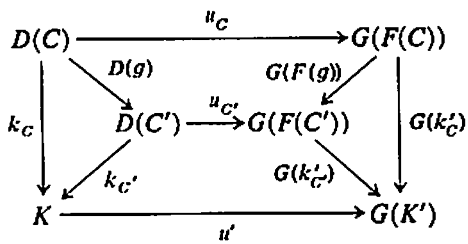
Since $((k_C), K)$ is a colimit, and thus an epi-sink, we have $f = G(\hat{f}) \circ u'$. The uniqueness of $\hat{f}$ with respect to this property follows from its construction and the fact that $(u_C, F(C))$, being a universal map, G-generates $F(C)$. Hence (u', K') is a G-universal map for K.

(ii) Suppose that (u', K') is a G-universal map for K. Since each $(u_C, F(C))$ is a G-universal map for $D(C)$, we have for each $C \in Ob(\mathscr{C})$ the existence of some $k'_C\colon F(C) \to K'$ such that the square

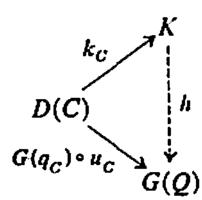


commutes.

We wish to show that $((k'_C), K')$ is a colimit of F. Since $((k_C), K)$ is a colimit of D and since $u: D \to G \circ F$ is a natural transformation, for each $g: C \to C'$ we have commutativity of all (except at most the right-hand triangle) of the diagram

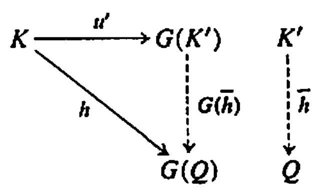


Since u_C G-generates $F(C)$, it follows that $k'_C = k'_{C'} \circ F(g)$. Thus $((k'_C), K')$ is a natural sink for F. To show that it is a colimit, assume that $((q_C), Q)$ is also a natural sink for F. It is easy to see that $((G(q_C) \circ u_C), G(Q))$ is then a natural sink for D, so that there is a unique morphism $h: K \to G(Q)$ such that each triangle



commutes.

Since (u', K') is a G-universal map for K, there is a unique $\bar{h}: K' \to Q$ such that the triangle



commutes.

But each of q_C and $\bar{h} \circ k'_C$ is a morphism x such that the triangle

$$\begin{array}{ccc} D(C) & \xrightarrow{u_C} & G(F(C)) \\ & \searrow^{h \circ k_C = G(q_C) \circ u_C} & \Big\downarrow G(x) \\ & & G(Q) \end{array} \qquad \begin{array}{c} F(C) \\ \Big\downarrow x \\ Q \end{array}$$

commutes.

Hence, since u_C G-generates $F(C)$, $q_C = \bar{h} \circ k'_C$ for each $C \in Ob(\mathscr{C})$. Since (k_C, K) is an epi-sink and (u', K') is a G-universal map, it is easy to see that $\bar{h}$ is unique with respect to this property. Consequently (k'_C, K') is a colimit of F. □

26.10 COROLLARY

Let $\mathscr{A}$ be a $\mathscr{C}$-cocomplete category and let $G: \mathscr{A} \to \mathscr{B}$. Then the full subcategory $\mathscr{B}'$ consisting of all objects of $\mathscr{B}$ that have universal maps with respect to G, is closed under the formation of $\mathscr{C}$-colimits in $\mathscr{B}$; i.e., if $D: \mathscr{C} \to \mathscr{B}$ has a colimit $((k_C), K)$ in $\mathscr{B}$ and if each $D(C)$ is in $\mathscr{B}'$, then K must be in $\mathscr{B}'$. □

26.11 THEOREM

Let $G: \mathscr{A} \to \mathscr{B}$ be a functor such that for each $B \in Ob(\mathscr{B})$ there exists a G-universal map (η_B, A_B).

(1) *Then there exists a unique functor $F: \mathscr{B} \to \mathscr{A}$ such that*
  (*i*) *for each $B \in Ob(\mathscr{B})$, $F(B) = A_B$; and*
  (*ii*) *$\eta = (\eta_B): 1_{\mathscr{B}} \to G \circ F$ is a natural transformation.*

(2) *Moreover, F preserves $\mathscr{C}$-colimits for each category $\mathscr{C}$, and*

(3) *There is a unique natural transformation $\varepsilon: F \circ G \to 1_{\mathscr{A}}$ such that*

$$G \xrightarrow{\eta * G} G \circ F \circ G \xrightarrow{G * \varepsilon} G = G \xrightarrow{1_G} G,$$

and

$$F \xrightarrow{F * \eta} F \circ G \circ F \xrightarrow{\varepsilon * F} F = F \xrightarrow{1_F} F;$$

i.e., for each $A \in Ob(\mathscr{A})$,

$$G(\varepsilon_A) \circ \eta_{G(A)} = 1_{G(A)},$$

and for each $B \in Ob(\mathscr{B})$,

$$\varepsilon_{F(B)} \circ F(\eta_B) = 1_{F(B)}.$$

Proof:

(*1*). This follows immediately from the preceding lemma (26.9), where D is the identity functor $1_{\mathscr{B}}: \mathscr{B} \to \mathscr{B}$.

(*2*). Suppose that $D: \mathscr{C} \to \mathscr{B}$ is a functor with colimit $((k_C), K)$. Since $K \in Ob(\mathscr{B})$, by hypothesis there is a G-universal map (η_K, A_K) for K. Thus by

Lemma 26.9, we have for each $C \in Ob(\mathscr{C})$ a morphism $k'_C\colon F \circ D(C) \to A_K$ such that $((k'_C), A_K)$ is a colimit of $F \circ D$ and each square

$$\begin{array}{ccccc} D(C) & \xrightarrow{\eta_{D(C)}} & G(F \circ D(C)) & & F \circ D(C) \\ {\scriptstyle k_C}\downarrow & & \vdots\, {\scriptstyle G(k'_C)} & & \vdots\, {\scriptstyle k'_C} \\ K & \xrightarrow[\eta_K]{} & G(A_K) = G(F(K)) & & A_K = F(K) \end{array}$$

commutes.

However, since $\eta\colon 1_{\mathscr{B}} \to G \circ F$ is a natural transformation, the morphisms $F(k_C)\colon F \circ D(C) \to F(K)$ also make the above squares commute. Since $\eta_{D(C)}$ G-generates $F \circ D(C)$, it follows that $F(k_C) = k'_C$, for each C, so that

$$((k'_C), A_K) = ((F(k_C)), F(K))$$

is a colimit of $F \circ D$. Hence F preserves $\mathscr{C}$-colimits.

(3). For each $A \in Ob(\mathscr{A})$, $G(A) \in Ob(\mathscr{B})$, so that $(\eta_{G(A)}, F(G(A)))$ is a G-universal map for $G(A)$. Consequently, by the definition of universal map, there exists a unique morphism $\varepsilon_A\colon F \circ G(A) \to A$ such that the triangle

$$\begin{array}{ccccc} G(A) & \xrightarrow{\eta_{G(A)}} & G(F \circ G(A)) & & F \circ G(A) \\ & {\scriptstyle 1_{G(A)}} \searrow & \vdots\, {\scriptstyle G(\varepsilon_A)} & & \vdots\, {\scriptstyle \varepsilon_A} \\ & & G(A) & & A \end{array}$$

commutes.

If $\varepsilon = (\varepsilon_A)$ is a natural transformation, then the commutativity of the triangle for each A clearly shows that

$$1_G = (G * \varepsilon) \circ (\eta * G) \qquad \text{(see 13.13).}$$

To show that ε is a natural transformation, let $f\colon A \to A'$. Then

$$\begin{aligned} G(f \circ \varepsilon_A) \circ \eta_{G(A)} &= G(f) \circ G(\varepsilon_A) \circ \eta_{G(A)} = G(f) \circ 1_{G(A)} \\ &= 1_{G(A')} \circ G(f) = G(\varepsilon_{A'}) \circ \eta_{G(A')} \circ G(f). \end{aligned}$$

But since $\eta\colon 1_{\mathscr{B}} \to G \circ F$ is a natural transformation, this becomes:

$$\begin{aligned} G(f \circ \varepsilon_A) \circ \eta_{G(A)} &= G(\varepsilon_{A'}) \circ (G \circ F)(G(f)) \circ \eta_{G(A)} \\ &= G(\varepsilon_{A'} \circ (F \circ G)(f)) \circ \eta_{G(A)}. \end{aligned}$$

Thus since $\eta_{G(A)}$ G-generates $(F \circ G)(A)$, we have

$$f \circ \varepsilon_A = \varepsilon_{A'} \circ (F \circ G)(f),$$

so that $\varepsilon = (\varepsilon_A)\colon 1_{\mathscr{A}} \to F \circ G$ is a natural transformation. To show that $(\varepsilon * F) \circ (F * \eta) = 1_F$, notice that for each $B \in Ob(\mathscr{B})$

$$G(\varepsilon_{F(B)} \circ F(\eta_B)) \circ \eta_B = G(\varepsilon_{F(B)}) \circ (G \circ F(\eta_B) \circ \eta_B).$$

But since η is a natural transformation, this is

$$G(\varepsilon_{F(B)}) \circ \eta_{G \circ F(B)} \circ \eta_B = G(1_{F(B)}) \circ \eta_B.$$

Consequently, since for each B, η_B G-generates $F(B)$,

$$\varepsilon_{F(B)} \circ F(\eta_B) = 1_{F(B)}.$$

Hence

$$(\varepsilon * F) \circ (F * \eta) = 1_F. \quad \square$$

26.12 PROPOSITION

Let $G: \mathscr{A} \to \mathscr{B}$ be a functor such that for each $B \in Ob(\mathscr{B})$ each of (η_B, A_B) and $(\hat{\eta}_B, \hat{A}_B)$ is a G-universal map for B. If F and $\hat{F}$ are the corresponding functors, guaranteed by the above theorem, such that $F(B) = A_B$ and $\hat{F}(B) = \hat{A}_B$, then F is naturally isomorphic to $\hat{F}$.

Proof: This follows immediately from the fact that universal maps are essentially unique (26.7). $\square$

EXERCISES

26A. Show how the second example in (3) of 26.2 is implied by Theorem D of Chapter I. [Note that each morphism $f: B \to Hom(C, D)$ induces a bilinear map from $B \times C$ to D.]

26B. For each of the examples of 26.2 with which you are familiar, determine the unique induced colimit-preserving functor F guaranteed by Theorem 26.11.

26C. Show that every set has a universal map with respect to the "squaring functor" (9A(b)). Describe the induced functor guaranteed by Theorem 26.11.

26D. If $\overline{\mathbf{Rng}}$ is the category of rings that do not necessarily have identities and $G: \mathbf{Rng} \to \overline{\mathbf{Rng}}$ is the forgetful functor (which forgets the identity), show that each object of $\overline{\mathbf{Rng}}$ has a G-universal map. What is the induced functor guaranteed by Theorem 26.11?

26E. Determine the "free lattice with two generators", i.e., the U-universal map (u, A_B) with respect to B, where U: **Lat** $\to$ **Set** is the forgetful functor and B is a two-element set.

26F. Determine whether or not the category of algebraically closed commutative fields is a reflective subcategory of the category of commutative fields.

26G. State the duals [including the contravariant duals (12.4)] of Theorem 26.11. Use the covariant dual to show that if $G: \mathscr{A} \to \mathscr{B}$ is a functor such that every $\mathscr{B}$-object has a G-universal map, then G preserves limits.

26H. *Universal Maps and Initial Objects*
Let $G: \mathscr{A} \to \mathscr{B}$ and $B \in Ob(\mathscr{B})$.

(a) If B is an initial object of $\mathscr{B}$ and u is the unique $\mathscr{B}$-morphism from B to $G(A)$, prove that the following are equivalent:

(i) (u, A) is a G-universal map for B.

(ii) A is an initial object of $\mathscr{A}$.

(b) If $\hat{B}: 1 \to \mathscr{B}$ is the functor whose value at the single object $*$ is B, prove that the following are equivalent:

(i) (u, A) is a G-universal map for B.

(ii) $(*, u, A)$ is an initial object of the comma category $(\hat{B}, G)$ (see 20D).

(c) Use part (b) to obtain an alternate proof that universal maps are "essentially unique" (see 26.7).

26I. *Colimits as Universal Maps*

Let $\mathscr{A}$ and I be categories, let $G: \mathscr{A} \to \mathscr{A}^I$ be the "constant functor" functor from $\mathscr{A}$ to $\mathscr{A}^I$ (15.8), let $D: I \to \mathscr{A}$ be a functor, and for each $i \in Ob(I)$ let

$$u_i: D(i) \to G(A)(i) = A$$

be an $\mathscr{A}$-morphism. Prove that:

(a) $u = (u_i): D \to G(A)$ is a natural transformation if and only if $((u_i), A)$ is a natural sink for D.

(b) (u, A) is a G-universal map if and only if $((u_i), A)$ is a colimit of D.

Now let $u = (u_i): D \to G(A)$ be a natural transformation. Prove that:

(c) u G-generates A if and only if $((u_i), A)$ is an epi-sink.

(d) u extremally G-generates A if and only if $((u_i), A)$ is an (extremal epi)-sink.

(e) Using Proposition 26.6 obtain a new proof that colimits are (extremal epi)-sinks (20.4 dual).

(f) Using Theorem 26.11 obtain a new proof of the commutation of colimits (25.4 dual).

(g) Dualize each of the above results (a)-(f).

§27 ADJOINT FUNCTORS

Recall that when we were considering what it should mean for a functor to "have an inverse", our first notion was that of an isomorphism between categories. However, since functors with such a "strong" type of inverse rarely occur, a more suitable notion was found in the conjunction of the notions full, faithful, and dense (i.e., equivalences between categories). Equivalences (although "weaker" than isomorphisms) have been seen to occur relatively frequently (14.16 and 14.18) and to be of such "strength" that they preserve and reflect categorical properties.

An even weaker notion of a functor which in some sense has an inverse, is that of a functor that has a left or right adjoint. Indeed, there is a plethora of such functors, yet the notion is still "strong" enough to be intensely interesting.

In the last section we have seen that if $G: \mathscr{A} \to \mathscr{B}$ is a functor such that each $\mathscr{B}$-object has a G-universal map, then there exists a functor $F: \mathscr{B} \to \mathscr{A}$ together with two natural transformations η and ε such that

$$(G * \varepsilon) \circ (\eta * G) = 1_G$$

and

$$(\varepsilon * F) \circ (F * \eta) = 1_F.$$

This section is devoted to a further study of such situations.

27.1 DEFINITION

(1) If $\mathscr{A}$ and $\mathscr{B}$ are categories, G and F are functors and η and ε are natural transformations such that:

(i) $G: \mathscr{A} \to \mathscr{B}$ and $F: \mathscr{B} \to \mathscr{A}$,

(ii) $\eta: 1_{\mathscr{B}} \to G \circ F$ and $\varepsilon: F \circ G \to 1_{\mathscr{A}}$,

(iii) $G \xrightarrow{\eta * G} G \circ F \circ G \xrightarrow{G * \varepsilon} G = G \xrightarrow{1_G} G$, and

$F \xrightarrow{F * \eta} F \circ G \circ F \xrightarrow{\varepsilon * F} F = F \xrightarrow{1_F} F$;

then this is called an **adjunction** or **adjoint situation**, and is denoted by

$$(\eta, \varepsilon): F \dashv G: (\mathscr{A}, \mathscr{B})$$

or more briefly by

$$(\eta, \varepsilon): F \dashv G$$

or simply by

$$F \dashv G.$$

(2) If $(\eta, \varepsilon): F \dashv G$, then F is said to be a **left adjoint of** G, G is said to be a **right adjoint of** F, η is called the **unit** of the adjunction and ε is called the **counit** of the adjunction.

(3) A functor $G: \mathscr{A} \to \mathscr{B}$ is said to **have a left adjoint** provided that there exist F, η, and ε such that $(\eta, \varepsilon): F \dashv G$. Similarly $F: \mathscr{B} \to \mathscr{A}$ is said to **have a right adjoint** provided that there are G, η, and ε such that $(\eta, \varepsilon): F \dashv G$. In other words, a functor has a left adjoint provided that it is the right adjoint of some functor, and it has a right adjoint provided that it is the left adjoint of some functor.

The next proposition follows immediately from the definition of equivalence situations (14.12).

27.2 PROPOSITION

(1) *If $(F, G, \eta, \varepsilon)$ is an equivalence situation, then*

$$(\eta, \varepsilon): F \dashv G$$

and

$$(\varepsilon^{-1}, \eta^{-1}): G \dashv F.$$

(2) *If $G: \mathscr{A} \to \mathscr{B}$ is an equivalence, then there exists a functor $F: \mathscr{B} \to \mathscr{A}$ that is simultaneously a left adjoint and a right adjoint of G.* □

Before giving some other concrete examples of adjoint situations, we wish to clarify the relationship between adjoint situations and universal maps. According to Theorem 26.11, whenever a functor $G: \mathscr{A} \to \mathscr{B}$ has the property that each $\mathscr{B}$-object has a G-universal map, this gives rise to an adjoint situation. The next theorem shows that each adjoint situation can actually be obtained in this way.

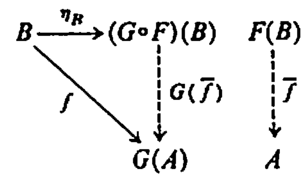
27.3 THEOREM

Let $G: \mathscr{A} \to \mathscr{B}$.

(1) *If each* $B \in Ob(\mathscr{B})$ *has a G-universal map* (η_B, A_B), *then there exists a unique adjoint situation* $(\eta, \varepsilon): F \dashv G$ *such that* $\eta = (\eta_B)$ *and for each* $B \in Ob(\mathscr{B})$, $F(B) = A_B$.

(2) *Conversely, if we have an adjoint situation* $(\eta, \varepsilon): F \dashv G$, *then for each* $B \in Ob(\mathscr{B})$, $(\eta_B, F(B))$ *is a G-universal map for B.*

Proof: The first assertion follows immediately from Theorem 26.11 and the definition of adjoint situation (27.1). To show (2), suppose that $B \in Ob(\mathscr{B})$ and $f: B \to G(A)$. We wish to find a unique morphism $\bar{f}: F(B) \to A$ such that the triangle



commutes.

Let $\bar{f} = \varepsilon_A \circ F(f)$. Then

$$G(\bar{f}) \circ \eta_B = G(\varepsilon_A) \circ (G \circ F)(f) \circ \eta_B.$$

But since $\eta: 1_B \to G \circ F$ is a natural transformation and since

$$(G * \varepsilon) \circ (\eta * G) = 1_G,$$

this becomes

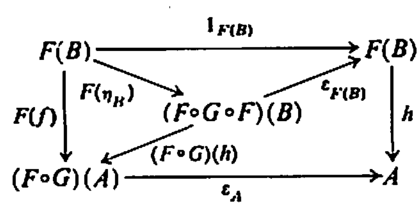
$$G(\varepsilon_A) \circ \eta_{G(A)} \circ f = 1_{G(A)} \circ f = f.$$

Hence $\bar{f}$ makes the triangle commute.

To show uniqueness, suppose that $h: F(B) \to A$ is a morphism with $f = G(h) \circ \eta_B$. Then since ε is a natural transformation and since

$$(\varepsilon * F) \circ (F * \eta) = 1_F,$$

we have commutativity of the diagram



Hence $h = \varepsilon_A \circ F(f)$. □

27.4 COROLLARY

If each of F *and* $\hat{F}$ *is a left adjoint of the functor* G, *then* F *is naturally isomorphic to* $\hat{F}$.

Proof: This is an immediate consequence of the above theorem and Proposition 26.12. □

27.5 EXAMPLES OF ADJOINT SITUATIONS

$\mathscr{A}$	$\mathscr{B}$	F (left adjoint)	G (right adjoint)	reference†
Grp	**Set**	free group functor	forgetful functor	26.2(1)a
R-**Mod**	**Set**	free R-module functor	forgetful functor	26.2(1)a
Top	**Set**	discrete space functor	forgetful functor	26.2(1)a
Rng	**Mon**	monoid ring functor	forgetful functor	26.2(1)b
Rng	**Ab**	tensor ring functor	forgetful functor	26.2(1)b
complete uniform spaces	uniform spaces	uniform completion functor	inclusion functor	26.2(2)a
$\mathbf{CompT_2}$	$\mathbf{CRegT_2}$	Stone-Čech compactification functor	inclusion functor	26.2(2)a
$\mathbf{BanSp_1}$	**NLinSp**	completion functor	inclusion functor	26.2(2)a
cancellative abelian semigroups	**Ab**	group of quotients functor	inclusion functor	26.2(2)a
Field	integral domains (injective homomorphisms)	field of quotients functor	inclusion functor	26.2(2)a
Ab	**Grp**	abelianization functor	inclusion functor	26.2(2)b

† Note that either the existence of a universal map for each $B \in Ob(\mathscr{B})$ or the existence of a co-universal map for each $A \in Ob(\mathscr{A})$ gives rise to an adjoint situation.

$\mathscr{A}$	$\mathscr{B}$	F (left adjoint)	G (right adjoint)	reference†
Ab	abelian torsion groups	inclusion functor	torsion subgroup functor	26.2(4)
torsion free abelian groups	**Ab**	torsion free functor	inclusion functor	26.2(2)b
$\mathbf{Ab}^{(n)}$	**Ab**	functor which takes B to B/nB	inclusion functor	26.2(2)b
Ab	$\mathbf{Ab}^{(n)}$	inclusion functor	(subgroup of elements of order being a divisor of n)-functor	26.2(4)
locally convex linear topological spaces	**LinTop**	locally convex modification functor	inclusion functor	26.2(2)d
Top	compactly generated spaces	inclusion functor	compactly generated refinement functor	26.2(4)
R-**Mod** (where R is commutative)	R-**Mod**	$\_ \otimes_R C$	$Hom(C, \_)$	26.2(3)
pTop	**pTop**	suspension functor	loop space functor	26.2(3)
Grp	pointed arcwise connected spaces (with homotopy equivalence classes of maps)	fundamental group functor Π_1	Eilenberg-Mac Lane space functor	

† Note that either the existence of a universal map for each $B \in Ob(\mathscr{B})$ or the existence of a co-universal map for each $A \in Ob(\mathscr{A})$ gives rise to an adjoint situation.

$\mathscr{A}$	$\mathscr{B}$	F (left adjoint)	G (right adjoint)	reference†
$\mathscr{A}$ (where $\mathscr{A}$ has I-colimits)	$\mathscr{A}^I$	$\underset{I}{Colim}$	"constant functor" functor	26I
$\mathscr{A}^I$	$\mathscr{A}$ (where $\mathscr{A}$ has I-limits)	"constant functor" functor	$\underset{I}{Lim}$	26I

† Note that either the existence of a universal map for each $B \in Ob(\mathscr{B})$ or the existence of a co-universal map for each $A \in Ob(\mathscr{A})$ gives rise to an adjoint situation.

27.6 THEOREM (DUALITY)

Let $G: \mathscr{A} \to \mathscr{B}$, $F: \mathscr{B} \to \mathscr{A}$, $\eta: 1_{\mathscr{B}} \to G \circ F$, and $\varepsilon: F \circ G \to 1_{\mathscr{A}}$. Then the following are equivalent:

(1) $(\eta, \varepsilon): F \dashv G: (\mathscr{A}, \mathscr{B})$.

(2) $(\varepsilon, \eta): G^{op} \dashv F^{op}: (\mathscr{B}^{op}, \mathscr{A}^{op})$.

Proof: If we are given the situation (1), then translating this into a statement in terms of $\mathscr{A}^{op}$ and $\mathscr{B}^{op}$, we see that $G^{op}: \mathscr{A}^{op} \to \mathscr{B}^{op}$, and $F^{op}: \mathscr{B}^{op} \to \mathscr{A}^{op}$ (see 9.2(5)). Since $\eta: 1_{\mathscr{B}} \to G \circ F$ is a morphism in $[\mathscr{B}, \mathscr{B}]$, it is also a morphism in

$$[\mathscr{B}, \mathscr{B}]^{op} \simeq [\mathscr{B}^{op}, \mathscr{B}^{op}] \qquad \text{(see 15A).}$$

Thus, in the category $[\mathscr{B}^{op}, \mathscr{B}^{op}]$, we have $\eta: G^{op} \circ F^{op} \to 1_{\mathscr{B}^{op}}$. Similarly $\varepsilon: 1_{\mathscr{A}^{op}} \to F^{op} \circ G^{op}$. Again, translating the statement:

$$G \xrightarrow{\eta * G} G \circ F \circ G \xrightarrow{G * \varepsilon} G = G \xrightarrow{1_G} G$$

from $[\mathscr{A}, \mathscr{B}]$ to $[\mathscr{A}^{op}, \mathscr{B}^{op}]$ means reversing the arrows; viz.

$$G^{op} \xleftarrow{\eta * G^{op}} G^{op} \circ F^{op} \circ G^{op} \xleftarrow{G^{op} * \varepsilon} G^{op} = G^{op} \xleftarrow{1_{G^{op}}} G^{op}.$$

Likewise, we have

$$F^{op} \xleftarrow{F^{op} * \eta} F^{op} \circ G^{op} \circ F^{op} \xleftarrow{\varepsilon * F^{op}} F^{op} = F^{op} \xleftarrow{1_{F^{op}}} F^{op}.$$

Thus we have the adjoint situation

$$(\varepsilon, \eta): G^{op} \dashv F^{op}: (\mathscr{B}^{op}, \mathscr{A}^{op}).$$

Clearly, applying the same considerations to situation (2) yields situation (1). □

27.7 THEOREM

If we have an adjoint situation $(\eta, \varepsilon): F \dashv G: (\mathscr{A}, \mathscr{B})$, then F preserves colimits and G preserves limits.

Proof: By Theorem 27.3 we know that for each $B \in Ob(\mathscr{B})$, $(\eta_B, F(B))$ is a G-universal map. Hence by Theorem 26.11, F preserves colimits. By duality (27.6), $(\varepsilon, \eta): G^{op} \dashv F^{op}$ is an adjoint situation, so that, by the above, G^{op} preserves colimits: i.e., G preserves limits. □

27.8 PROPOSITION

Adjoint situations can be composed; specifically if

$$(\eta, \varepsilon): F \dashv G: (\mathscr{A}, \mathscr{B})$$

and

$$(\nu, \delta): S \dashv T: (\mathscr{B}, \mathscr{C}),$$

then

$$((T * \eta * S) \circ \nu, \varepsilon \circ (F * \delta * G)): F \circ S \dashv T \circ G: (\mathscr{A}, \mathscr{C}).$$

Proof: This is a straightforward application of Godement's "five" rules (13.14). □

27.9 THEOREM

If $G: \mathscr{A} \to \mathscr{B}$ *and* $F: \mathscr{B} \to \mathscr{A}$, then the following are equivalent:

(1) *F is a left adjoint of G; i.e., there exist natural transformations* η *and* ε *such that* $(\eta, \varepsilon): F \dashv G$.

(2) *The associated set-valued bifunctors* (10B)

$$hom(F\_\_, \_\_): \mathscr{B}^{op} \times \mathscr{A} \to \mathbf{Set},$$

and

$$hom(\_\_, G\_\_): \mathscr{B}^{op} \times \mathscr{A} \to \mathbf{Set}$$

are naturally isomorphic.

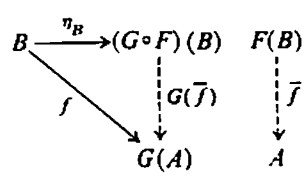
Proof:

(1) ⇒ *(2)*. If $(\eta, \varepsilon): F \dashv G$, then define $\alpha: hom(F\_\_, \_\_) \to hom(\_\_, G\_\_)$ by

$$\alpha_{BA}(f) = G(f) \circ \eta_B.$$

If $\alpha_{BA}(f) = \alpha_{BA}(g)$, then $G(f) \circ \eta_B = G(g) \circ \eta_B$, so that since $(\eta_B, F(B))$ is a G-universal map for B, by the uniqueness condition for universal maps, $f = g$. Consequently, α_{BA} is an injective function.

If $f: B \to G(A)$, then again by the property of universal maps there is some morphism $\bar{f}: F(B) \to A$ such that the triangle



commutes, i.e., such that $\alpha_{BA}(\bar{f}) = G(\bar{f}) \circ \eta_B = f$. Consequently α_{BA} is surjective; hence, bijective. Thus to establish that α is a natural transformation, we need only show that if

$$B' \xrightarrow{g} B \qquad \text{and} \qquad A \xrightarrow{f} A'$$

then the square

$$\begin{array}{ccc} hom(F(B), A) & \xrightarrow{\alpha_{BA}} & hom(B, G(A)) \\ {\scriptstyle hom(F(g), f)}\downarrow & & \downarrow{\scriptstyle hom(g, G(f))} \\ hom(F(B'), A') & \xrightarrow[\alpha_{B'A'}]{} & hom(B', G(A')) \end{array}$$

commutes.

Let $x \in hom(F(B), A)$. Then

$$\begin{aligned}(\alpha_{B'A'} \circ hom(F(g), f))(x) &= \alpha_{B'A'}(f \circ x \circ F(g)) \\ &= G(f \circ x \circ F(g)) \circ \eta_{B'} = G(f \circ x) \circ (G \circ F)(g) \circ \eta_{B'}.\end{aligned}$$

But since $\eta: 1_{\mathscr{B}} \to G \circ F$ is a natural transformation, this is

$$\begin{aligned}G(f \circ x) \circ \eta_B \circ g &= G(f) \circ (G(x) \circ \eta_B) \circ g = hom(g, G(f))(G(x) \circ \eta_B) \\ &= (hom(g, G(f)) \circ \alpha_{BA})(x).\end{aligned}$$

Thus the square commutes.

(2) $\Rightarrow$ *(1)*. Suppose that $\alpha: hom(F\_\_, \_\_) \to hom(\_\_, G\_\_)$ is a natural isomorphism. For each $B \in Ob(\mathscr{B})$, let

$$\eta_B = \alpha_{B,F(B)}(1_{F(B)}).$$

By Theorem 27.3, we need only show that for each $B \in Ob(\mathscr{B})$, $(\eta_B, F(B))$ is a G-universal map for B.

Suppose that $f: B \to G(A)$. Applying the commutativity of the square

$$\begin{array}{ccc} hom(F(B), F(B)) & \xrightarrow{\alpha_{B,F(B)}} & hom(B, G \circ F(B)) \\ {\scriptstyle hom(1_{F(B)}, \alpha^{-1}_{BA}(f))}\downarrow & & \downarrow{\scriptstyle hom(1_B, G(\alpha^{-1}_{BA}(f)))} \\ hom(F(B), A) & \xrightarrow[\alpha_{BA}]{} & hom(B, G(A)) \end{array}$$

to the element $1_{F(B)} \in hom(F(B), F(B))$, we have

$$\begin{aligned}f = \alpha_{BA}(\alpha^{-1}_{BA}(f)) &= (\alpha_{BA} \circ hom(1_{F(B)}, \alpha^{-1}_{BA}(f)))(1_{F(B)}) \\ &= (hom(1_B, G(\alpha^{-1}_{BA}(f)) \circ \alpha_{B,F(B)})(1_{F(B)}) = G(\alpha^{-1}_{BA}(f)) \circ \eta_B.\end{aligned}$$

Hence, the triangle

$$\begin{array}{ccccc} B & \xrightarrow{\eta_B} & (G \circ F)(B) & \quad & F(B) \\ & {\searrow}^{f} & \downarrow{\scriptstyle G(\alpha^{-1}_{BA}(f))} & & \downarrow{\scriptstyle \alpha^{-1}_{BA}(f)} \\ & & G(A) & & A \end{array}$$

commutes.

To show uniqueness, suppose that $g: F(B) \to A$ such that $G(g) \circ \eta_B = f$. Again by commutativity of the above square, with $\alpha_{BA}^{-1}(f)$ replaced by g, we have

$$\begin{aligned}\alpha_{BA}(g) &= \alpha_{BA}(hom(1_{F(B)}, g)(1_{F(B)})) \\ &= (hom(1_B, G(g)) \circ \alpha_{B,F(B)})(1_{F(B)}) = G(g) \circ \eta_B = f.\end{aligned}$$

Thus, since α_{BA} is bijective, $g = \alpha_{BA}^{-1}(f)$. □

This theorem, together with earlier ones, tells us that if $G: \mathscr{A} \to \mathscr{B}$ and $F: \mathscr{B} \to \mathscr{A}$, then there are at least four ways of describing the fact that F is a left adjoint of G:

(1) by means of a family $(\eta_B, F(B))$ of universal maps;

(2) by means of a family $(G(A), \varepsilon_A)$ of co-universal maps;

(3) by means of natural transformations

$$\eta: 1_{\mathscr{B}} \to G \circ F \qquad \text{and} \qquad \varepsilon: F \circ G \to 1_{\mathscr{A}}$$

such that

$$G \xrightarrow{\eta * G} G \circ F \circ G \xrightarrow{G * \varepsilon} G = G \xrightarrow{1_G} G,$$

and

$$F \xrightarrow{F * \eta} F \circ G \circ F \xrightarrow{\varepsilon * F} F = F \xrightarrow{1_F} F;$$

(4) by means of a natural isomorphism

$$\alpha = (\alpha_{BA}): hom(F\_\_, \_\_) \longrightarrow hom(\_\_, G\_\_).$$

The fourth way is perhaps the quickest and easiest to state and is thus often used as the definition of adjoint situation. However, in practice, the first or second way is often easier to establish and closer to one's intuition.

EXERCISES

27A. Let H and K be groups considered as one-element categories, and let $f: H \to K$ be a group homomorphism (i.e., a functor from H to K). Show that the following are equivalent:

(a) f is an isomorphism.
(b) f has a left adjoint.
(c) f has a right adjoint.

27B. Let A and B be classes considered as discrete categories, and let $f: A \to B$ be a function (i.e., a functor from A to B). Show that the following are equivalent:

(a) f is a bijective function.
(b) f is an equivalence.
(c) f is an isomorphism.
(d) f has a left adjoint.
(e) f has a right adjoint.

27C. Let A and B be sets and let $f: A \to B$ be any function. Since the power sets $\mathscr{P}(A)$ and $\mathscr{P}(B)$ are partially-ordered by inclusion, they can be considered as categories (3.5(6)). Show that:

(a) the induced functions

$$f[\quad]: \mathscr{P}(A) \to \mathscr{P}(B)$$

and

$$f^{-1}[\quad]: \mathscr{P}(B) \to \mathscr{P}(A)$$

are functors.
(b) $f[\quad]$ is a left adjoint of $f^{-1}[\quad]$.
(c) Using the fact that right adjoints preserve limits, conclude that

$$f^{-1}[\cap C_i] = \cap f^{-1}[C_i]$$

for any family (C_i) of subsets of B.

27D. Let A be a totally-ordered set and let $\mathbf{R}$ be the real numbers (considered as a totally-ordered set). Consider each of A and $\mathbf{R}$ as categories (3.5(6)) and let $g: A \to \mathbf{R}$ be a functor (i.e., a monotone function).

(a) Show that if g has a left adjoint, then g is upper semi-continuous (where now A and $\mathbf{R}$ are thought of as ordered topological spaces).
(b) Show that if A is complete and g is upper semi-continuous, then g has a left adjoint.
(c) State analogous results involving lower semi-continuity.

27E. Let $\mathscr{C}$ be a category that has finite products and finite coproducts and let $G: \mathscr{C} \to \mathscr{C}$ be defined by:

$$G(A) = A \times A$$

$$G(f) = f \times f.$$

Let $F: \mathscr{C} \to \mathscr{C}$ be defined by:

$$F(A) = A \amalg A$$

$$F(f) = f \amalg f.$$

Show that F is a left adjoint of G.

27F. Suppose that $\mathscr{C}$ is a category that has coproducts and A is a fixed object in $\mathscr{C}$. Let F: **Set** $\to \mathscr{C}$ be the "copower functor" that assigns to each set I the object ${}^I A$ (i.e., the Ith copower of A) and to each function $f: I \to J$ the unique induced morphism $\bar{f}: {}^I A \to {}^J A$ (for which $\bar{f} \circ \mu_i = \mu_{f(i)}$, for each $i \in I$). Show that F is a left adjoint of $hom(A, \_): \mathscr{C} \to$ **Set**. (Cf. 27E.)

27G. Prove that every functor that has a left adjoint preserves monomorphisms.

27H. Given $(\eta, \varepsilon): F \dashv G: (\mathscr{A}, \mathscr{B})$. Show that:

(a) G is faithful iff all ε_A's are epimorphisms.

(b) G is full iff all ε_A's are sections.

(c) If all ε_A's are regular epimorphisms, then G reflects limits.

27I. Dualize statements (a), (b), and (c) in Exercise 27H.

27J. For each of the examples of 27.5 with which you are familiar, determine both the unit and the counit of the adjunction.

27K. Prove that if F is a left adjoint of G, $\hat{F}$ is a left adjoint of $\hat{G}$, and G is naturally isomorphic to $\hat{G}$, then F is naturally isomorphic to $\hat{F}$.

27L. Prove that if $(\eta, \varepsilon): F \dashv G$, then there exists a unique natural isomorphism

$$\alpha: hom(F\_, \_) \longrightarrow hom(\_, G\_)$$

such that for each $f: F(B) \to A$,

$$\alpha_{BA}(f) = G(f) \circ \eta_B$$

and for each $g: B \to G(A)$

$$\alpha_{BA}^{-1}(g) = \varepsilon_A \circ F(g).$$

27M. Prove that if $F: \mathscr{B} \to \mathscr{A}$ and $G: \mathscr{A} \to \mathscr{B}$ are functors and

$$\alpha: hom(F\_, \_) \longrightarrow hom(\_, G\_)$$

is a natural isomorphism, then there is a unique adjoint situation $(\eta, \varepsilon): F \dashv G$ such that for each $B \in Ob(\mathscr{B})$

$$\eta_B = \alpha_{B,FB}(1_{FB})$$

and for each $A \in Ob(\mathscr{A})$

$$\varepsilon_A = \alpha_{GA,A}^{-1}(1_{GA}).$$

27N. Suppose that $\mathscr{A} \xrightarrow{G} \mathscr{B}$ and $\mathscr{B} \xrightarrow{H} \mathscr{A}$ are functors and $\xi: 1_{\mathscr{B}} \to G \circ H$ and $\tau: H \circ G \to 1_{\mathscr{A}}$ are natural transformations such that

$$G \xrightarrow{\xi * G} G \circ H \circ G \xrightarrow{G * \tau} G = G \xrightarrow{1_G} G.$$

(a) Show that G does not necessarily have a left adjoint.

(b) Show that if $\mathscr{B}$ has any one of the properties:

(i) $\mathscr{B}$ has equalizers,

(ii) $\mathscr{B}$ has coequalizers,

(iii) idempotents in $\mathscr{B}$ split (17D),

then G does have a left adjoint.

27O. Show that if $\mathscr{A} \xrightarrow{G} \mathscr{B}$ and $\mathscr{B} \xrightarrow{F} \mathscr{A}$, then F is a left adjoint of G if and only if there is an isomorphism between the comma categories $(F, 1_{\mathscr{A}})$ and $(1_{\mathscr{B}}, G)$ that commutes with the projection functors to $\mathscr{A}$ and $\mathscr{B}$ (see 20D(e)).

27P. *Adjoints and Functor Categories*
(a) Prove that for any category I, the internal *hom*-functor

$$Hom(I, \_\_): \mathscr{CAT} \longrightarrow \mathscr{CAT}$$

(15.2) preserves adjoint situations; i.e., if

$$F \dashv G: (\mathscr{A}, \mathscr{B}), \quad \text{then} \quad (F \circ \_\_) \dashv (G \circ \_\_): (\mathscr{A}^I, \mathscr{B}^I).$$

(b) State and prove a similar assertion for the contravariant internal *hom*-functor $Hom(\_\_, I)$.
(c) Suppose that $F \dashv G: (\mathscr{A}, \mathscr{B})$; I is a category; $C_{\mathscr{A}}: \mathscr{A} \to \mathscr{A}^I$ and $C_{\mathscr{B}}: \mathscr{B} \to \mathscr{B}^I$ are "constant functor" functors (15.8), and $\underset{I}{Lim}_{\mathscr{A}}: \mathscr{A}^I \to \mathscr{A}$ and $\underset{I}{Lim}_{\mathscr{B}}: \mathscr{B}^I \to \mathscr{B}$ are the corresponding limit functors (23.4).
Show that

$$C_{\mathscr{A}} \circ F \dashv G \circ \underset{I}{Lim}_{\mathscr{A}}: (\mathscr{A}^I, \mathscr{B})$$

and that

$$(F \circ \_\_) \circ C_{\mathscr{B}} \dashv \underset{I}{Lim}_{\mathscr{B}} (G \circ \_\_): (\mathscr{A}^I, \mathscr{B})$$

and conclude that $G \circ \underset{I}{Lim}_{\mathscr{A}}$ and $\underset{I}{Lim}_{\mathscr{B}}(G \circ \_\_)$ are essentially the same; i.e., that right adjoints preserve I-limits.
(d) Using (a) and (c), obtain an alternate proof of the commutation of limits (cf. 25.4).

27Q. *Adjoints and Galois Correspondences*
(a) Suppose that $\mathscr{A} = (A, \prec)$ and $\mathscr{B} = (B, \leq)$ are quasi-ordered classes and $G: \mathscr{A} \to \mathscr{B}$ and $F: \mathscr{B} \to \mathscr{A}$ are order-reversing functions. Consider the following four statements about this situation:

(1) For all $a \in A$ and for all $b \in B$, $a \prec F(b)$ if and only if $b \leq G(a)$.

(2) For all $a \in A$, $a \prec F(G(a))$ and for all $b \in B$, $b \leq G(F(b))$.

(3) For all $a \in A$, $G(a) \approx G(F(G(a)))$ and for all $b \in B$, $F(G(F(b))) \approx F(b)$ (where in either quasi-ordered class $\approx$ is the equivalence relation induced by the quasi-order relation).

(4) If $\bar{\mathscr{A}} = \{F(b) \mid b \in B\}$ and $\bar{\mathscr{B}} = \{G(a) \mid a \in A\}$ (each with the induced order), then the restrictions $\bar{G}: \bar{\mathscr{A}} \to \bar{\mathscr{B}}$ and $\bar{F}: \bar{\mathscr{B}} \to \bar{\mathscr{A}}$ are (up to the equivalence $\approx$) anti-isomorphisms.

If condition (2) is satisfied, then the quadruple $(\mathscr{A}, \mathscr{B}, G, F)$ is called a **Galois correspondence**. Now regard $\mathscr{A}$ and $\mathscr{B}$ as categories (3.5(6)) and $G: \mathscr{A}^{op} \to \mathscr{B}$ and $F: \mathscr{B} \to \mathscr{A}^{op}$ as functors. With this interpretation, show that the four conditions above translate into the following four conditions:

(1) $F \dashv G: (\mathscr{A}^{op}, \mathscr{B})$; i.e., F is a left adjoint of G.

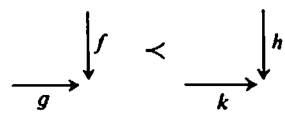
(2) There exist natural transformations $\varepsilon: F \circ G \to 1_{\mathscr{A}^{op}}$ and $\eta: 1_{\mathscr{B}} \to G \circ F$.

(3) There exist natural isomorphisms from G to $G \circ F \circ G$ and from F to $F \circ G \circ F$.

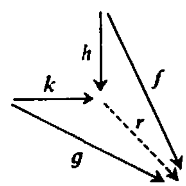
(4) $\bar{F}$ and $\bar{G}$ are equivalences.

(b) Show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

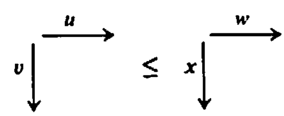
(c) Let $\mathscr{C}$ be a category that has pullbacks and pushouts, let A be the class of all pairs of morphisms in $\mathscr{C}$ with common codomain, with the quasi-order relation $\prec$ defined by:



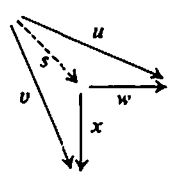
if and only if there is a morphism r such that the diagram



commutes,
and let B be the class of all pairs of morphisms in $\mathscr{C}$ with common domain, with the quasi-order relation $\leq$ defined by:



if and only if there is a morphism s such that the diagram



commutes.

Using pullbacks and pushouts, define functions $G\colon (A, \prec) \to (B, \leq)$ and $F\colon (B, \leq) \to (A, \prec)$ such that $((A, \prec), (B, \leq), G, F)$ is a Galois correspondence (cf. Exercise 21D). Obtain a similar result for sinks and sources indexed by a given class I.

27R. Show that if $\mathscr{C}$ is a pointed category that has kernels and cokernels and if $C \xrightarrow{f} D$ is a $\mathscr{C}$-morphism, then

(a) $Ker(Cok\, f) \leq (C, f)$;

(b) $(f, D) \geq Cok(Ker\, f)$;

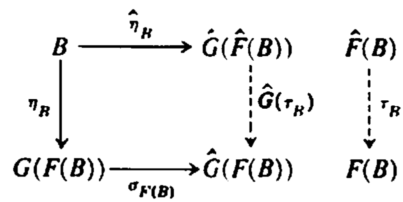
(c) $Ker(Cok(Ker\, f)) \approx Ker\, f$; and

(d) $Cok(Ker(Cok\, f)) \approx Cok\, f$.

[*Hint:* Use Exercise 27Q.]

27S. Suppose that $(\eta, \varepsilon)\colon F \dashv G\colon (\mathscr{A}, \mathscr{B})$, $(\hat{\eta}, \hat{\varepsilon})\colon \hat{F} \dashv \hat{G}\colon (\mathscr{A}, \mathscr{B})$, and $\sigma\colon G \to \hat{G}$ is a natural transformation.

(a) Show that there is a unique natural transformation $\tau\colon \hat{F} \to F$ such that for each $B \in Ob(\mathscr{B})$, the square



commutes.

(τ is often called the **conjugate of** σ.)

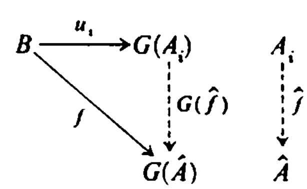
(b) Use part (a) to obtain an alternate proof of the essential uniqueness of left adjoints (27.4).

§28 EXISTENCE OF ADJOINTS

In this section we will see that under suitable conditions on $\mathscr{A}$ one is able to characterize those functors $F: \mathscr{A} \to \mathscr{B}$ that have a left adjoint (and by duality, those which have a right adjoint). We have seen before that category theory consists essentially of both general constructions and smallness considerations. The adjoint functor theorems of this section are prime examples of this fact. Below we will see that each of them contains a completeness condition and a smallness condition, neither of which can be eliminated.

28.1 DEFINITION

Let $G: \mathscr{A} \to \mathscr{B}$ be a functor and let B be a $\mathscr{B}$-object. A set-indexed family $(u_i, A_i)_I$, where each A_i is an $\mathscr{A}$-object and each $u_i: B \to G(A_i)$ is a $\mathscr{B}$-morphism is called a G-**solution set for** B provided that for each $\mathscr{A}$-object $\hat{A}$ and each morphism $f: B \to G(\hat{A})$, there exists some $i \in I$ and some morphism $\hat{f}: A_i \to \hat{A}$ such that the triangle



commutes.

28.2 PROPOSITION

Let $G: \mathscr{A} \to \mathscr{B}$ be a functor and let $u: B \to G(A)$ be a $\mathscr{B}$-morphism. Then the following are equivalent:

(1) *(u, A) is a G-universal map for B.*

(2) *(u, A) is a G-solution set for B and u G-generates A.* □

28.3 FIRST ADJOINT FUNCTOR THEOREM

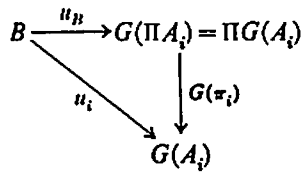
Let $\mathscr{A}$ be complete and $G: \mathscr{A} \to \mathscr{B}$. Then G has a left adjoint if and only if

(1) *G preserves limits, and*

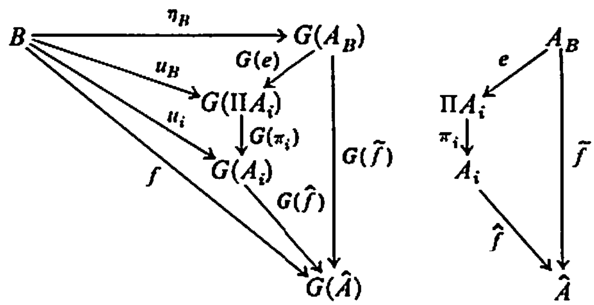
(2) *Each $\mathscr{B}$-object has a G-solution set.*

Proof: If $(\eta, \varepsilon): F \dashv G$, then by Theorem 27.7 G preserves limits, and by Theorem 27.3 for each $B \in Ob(\mathscr{B})$, $(\eta_B, F(B))$ is a G-universal map for B. Hence by the above proposition it is also a G-solution set for B.

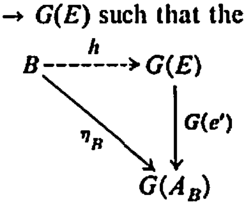
To show the converse, suppose that (1) and (2) are satisfied. By Theorem 27.3 it is sufficient to show that there exists a G-universal map for each $\mathscr{B}$-object B. Let $(u_i, A_i)_I$ be a G-solution set for B, and let $(\Pi A_i, \pi_i)$ be the product of the family $(A_i)_I$. Since G preserves products, it follows that $(G(\Pi A_i), G(\pi_i))$ is the product of the family $(G(A_i))_I$. Thus (by the definition of product) there exists a unique morphism $u_B: B \to G(\Pi A_i)$ such that for each $i \in I$ the triangle



commutes. Obviously $(u_B, \Pi A_i)$ is a G-solution set for B. Now let $(g_j)_J$ be the family of all morphisms in $hom(\Pi A_i, \Pi A_i)$ with the property that $G(g_j) \circ u_B = u_B$, and let (A_B, e) be the multiple equalizer of $(g_j)_J$ (16.10). Now since G preserves limits, $(G(A_B), G(e))$ is the multiple equalizer of the family $(G(g_j))_J$. Thus there is a morphism $\eta_B: B \to G(A_B)$ such that $G(e) \circ \eta_B = u_B$. We claim that (η_B, A_B) is a G-universal map for B. To show that (η_B, A_B) is a G-solution set for B, suppose that $\hat{A}$ is an $\mathscr{A}$-object and $f: B \to G(\hat{A})$. Since $(u_i, A_i)_I$ is a G-solution set for B, there is some $u_i: B \to G(A_i)$ and some $\hat{f}: A_i \to \hat{A}$ such that $G(\hat{f}) \circ u_i = f$. Let $\bar{f} = \hat{f} \circ \pi_i \circ e$; then $f = G(\bar{f}) \circ \eta_B$.



To show that η_B G-generates A_B, let $A_B \underset{s}{\overset{r}{\rightrightarrows}} \bar{A}$ be a pair of $\mathscr{A}$-morphisms such that $G(r) \circ \eta_B = G(s) \circ \eta_B$. Let (E, e') be the equalizer of r and s. Since G preserves limits, $(G(E), G(e'))$ is the equalizer of $G(r)$ and $G(s)$; so that there exists some morphism $h: B \to G(E)$ such that the triangle



commutes. Since $(u_B, \Pi A_i)$ is a G-solution set for B, there is some morphism $\bar{h}: \Pi A_i \to E$ such that $G(\bar{h}) \circ u_B = h$. Now $e \circ e' \circ \bar{h}$ is a morphism from ΠA_i to ΠA_i such that

$$G(e \circ e' \circ \bar{h}) \circ u_B = G(e) \circ G(e') \circ G(\bar{h}) \circ u_B = G(e) \circ G(e') \circ h = G(e) \circ \eta_B = u_B.$$

Consequently, $e \circ e' \circ \bar{h}$ belongs to the family $(g_j)_J$. Thus since (A_B, e) is the multiple equalizer of $(g_j)_J$ and since $1_{\Pi A_i}$ also belongs to $(g_j)_J$, we have

$$(e \circ e' \circ \bar{h}) \circ e = 1_{\Pi A_i} \circ e = e \circ 1_{A_B}.$$

The fact that e is a monomorphism (16.11) implies that $e' \circ \bar{h} \circ e = 1_{A_B}$. Thus e' is a retraction and a regular monomorphism, hence an isomorphism (6.7), so that $r = s$ (16.7). Consequently, η_B G-generates A_B, so that by the above proposition (η_B, A_B) is a G-universal map for B. □

Whereas the "limit preservation" condition in the above theorem is a "nice" condition that is usually easy to check in concrete cases, the "solution set" condition is cumbersome and usually difficult to check (especially since the theorem gives no idea of how to find a solution set). To help alleviate this problem, we will essentially split the latter condition into two conditions, the first being a "factorization" condition that is automatically satisfied in many cases (Lemma 28.6) and the second being another (unavoidable) "smallness" condition that is usually easier to check than the solution set condition. The latter smallness condition is essentially the existence of some "canonical solution set" (Theorem 28.9).

28.4 DEFINITION

Let $G: \mathscr{A} \to \mathscr{B}$ be a functor and $f: B \to G(A)$ be a $\mathscr{B}$-morphism. A factorization

$$B \xrightarrow{f} G(A) = B \xrightarrow{g} G(\bar{A}) \xrightarrow{G(m)} G(A)$$

of f is called an **(extremal G-generating, mono)-factorization of (f, A)** if and only if g extremally G-generates $\bar{A}$ and $m: \bar{A} \to A$ is an $\mathscr{A}$-monomorphism. Similarly one can define **(G-generating, extremal mono)-factorizations**.

28.5 EXAMPLES

(1) If $G: \mathscr{A} \to \mathscr{A}$ is the identity functor and $f: A \to B$ is an $\mathscr{A}$-morphism, then an (extremal G-generating, mono)-factorization of (f, B) is the same as an (extremal epi, mono)-factorization of f (cf. 17.15).

(2) If $G: \mathscr{A} \to \mathscr{A}^I$ is the "constant functor" functor (15.8) and $f = (f_i): B \to G(A)$ is an $\mathscr{A}^I$-morphism, then an (extremal G-generating, mono)-factorization of (f, A) is the same as an [(extremal epi)-sink, mono]-factorization of the sink $((f_i), A)$ (cf. 19.13).

(3) If U: **Grp** → **Set** is the forgetful functor and $f: B \to U(A)$ is a function, then an (extremal G-generating, mono)-factorization of (f, A) is (up to isomorphism) the factorization

$$B \xrightarrow{f} U(A) = B \xrightarrow{g} U(\bar{A}) \xrightarrow{U(m)} U(A)$$

where $\bar{A}$ is the subgroup of A generated by the set $f[B]$, $m: \bar{A} \to A$ is the inclusion homomorphism, and g is the unique function with $f = U(m) \circ g$.

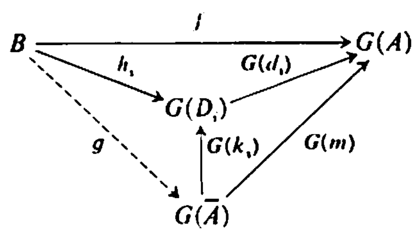
28.6 FACTORIZATION LEMMA

If $\mathscr{A}$ is well-powered, has intersections and equalizers, and the functor $G: \mathscr{A} \to \mathscr{B}$ preserves limits, then for each $\mathscr{B}$-morphism of the form $B \xrightarrow{f} G(A)$, the pair (f, A) has an (extremal G-generating, mono)-factorization.

Proof: Let $\mathscr{S}$ be the class $(D_i, d_i)_I$ of all subobjects of A that are part of some factorization

$$B \xrightarrow{f} G(A) = B \xrightarrow{h_i} G(D_i) \xrightarrow{G(d_i)} G(A)$$

of (f, A), and let $(\bar{A}, m)$ be the intersection of $\mathscr{S}$ (17.7), where for each i, $m = d_i \circ k_i$. Since G preserves limits $(G(\bar{A}), G(m))$ is the intersection of the family $(G(D_i), G(d_i))_I$. Thus there is a morphism $g: B \to G(\bar{A})$ such that the diagram



commutes for each $i \in I$.

We need only show that g extremally G-generates $\bar{A}$. Notice that by Proposition 26.5 it is sufficient to show that g satisfies the extremal condition; i.e., that if

$$B \xrightarrow{g} G(\bar{A}) = B \xrightarrow{g'} G(\bar{A}) \xrightarrow{G(h)} G(\bar{A})$$

where $h: \bar{A} \to \bar{A}$ is a monomorphism, then h must be an isomorphism. If indeed $g = G(h) \circ g'$, where h is a monomorphism, then $(\bar{A}, m \circ h)$ belongs to $\mathscr{S}$ so that $m \circ h = d_j$ for some $j \in I$. Thus we have

$$d_j \circ 1_{\bar{A}} = d_j = m \circ h = d_j \circ k_j \circ h$$

so that since d_j is a monomorphism, $1_{\bar{A}} = k_j \circ h$. Consequently h is both a monomorphism and a retraction, hence an isomorphism. ☐

28.7 COROLLARY

Every well-powered category that has intersections and equalizers is (extremal epi, mono)-factorizable (cf. 17.16). ☐

28.8 COROLLARY

If $\mathscr{A}$ is well-powered and has intersections and equalizers, then every sink in $\mathscr{A}$ has an [(extremal epi)-sink, mono]-factorization (cf. 19.14). ☐

28.9 THEOREM

If $\mathscr{A}$ is complete and well-powered, $G: \mathscr{A} \to \mathscr{B}$ preserves limits, and each $\mathscr{B}$-object extremally G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects, then G has a left adjoint.

Proof: Let B be a $\mathscr{B}$-object and let $(B \xrightarrow{g_i} G(A_i), A_i)_I$ be a representative set for the class of all pairs (g, A) for which $g: B \to G(A)$ extremally G-generates A. Then according to the Factorization Lemma (28.6), this is a G-solution set for B. Apply the First Adjoint Functor Theorem (28.3). □

28.10 SECOND ADJOINT FUNCTOR THEOREM

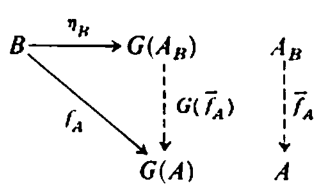
Let $\mathscr{A}$ be complete, well-powered and extremally co-(well-powered). Then $G: \mathscr{A} \to \mathscr{B}$ has a left adjoint if and only if the following two conditions are satisfied:

(1) *G preserves limits.*

(2) *Each B-object extremally G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects.*

Proof: That the two conditions are sufficient for the existence of a left adjoint has been shown in the preceding theorem (28.9). To show that they are necessary, first recall that each right adjoint preserves limits (27.7). Thus we need only verify that (2) holds.

Let B be a $\mathscr{B}$-object and suppose that $\mathscr{R}$ is a class of pairwise non-isomorphic objects of $\mathscr{A}$ such that for each $A \in \mathscr{R}$ there is some $f_A: B \to G(A)$ which extremally G-generates A. Since G has a left adjoint, there exists a universal map (η_B, A_B) for B (27.3). Hence for each $A \in \mathscr{R}$ there is a morphism $\bar{f}_A: A_B \to A$ such that the triangle



commutes.

We claim that each $\bar{f}_A$ is an extremal epimorphism. To show this, it suffices to show that if

$$A_B \xrightarrow{\bar{f}_A} A = A_B \xrightarrow{g} \hat{A} \xrightarrow{m} A$$

is a factorization of $\bar{f}_A$, where m is a monomorphism, then m must be an isomorphism (17.14). If indeed $\bar{f}_A = m \circ g$ where m is a monomorphism, then we have the following factorization of f_A:

$$B \xrightarrow{f_A} G(A) = B \xrightarrow{G(g)\cdot\eta_B} G(\bar{A}) \xrightarrow{G(m)} G(A).$$

But since f_A extremally generates A, this implies that m is an isomorphism. Consequently for each $A \in \mathscr{R}$, $(\bar{f}_A, A)$ is an extremal quotient object of A_B.

Thus since the objects in $\mathscr{R}$ are pairwise non-isomorphic and $\mathscr{A}$ is extremally co-(well-powered), $\mathscr{R}$ must be a set. □

It is interesting to observe that in general both of the smallness hypotheses of the Second Adjoint Functor Theorem are needed [for well-powered, see Exercise 28C; and for extremally co-(well-powered), see Exercise 28D]. The next theorem provides a remarkable simplification in the case where the category $\mathscr{A}$ has a coseparator. Recall that we have earlier seen that this condition (in conjunction with well-poweredness) is actually a quite strong smallness condition (23.14).

28.11 SPECIAL ADJOINT FUNCTOR THEOREM

Suppose that $\mathscr{A}$ is well-powered, complete, and has a coseparator C. Then for each functor $G: \mathscr{A} \to \mathscr{B}$ the following are equivalent:

(1) *G has a left adjoint.*

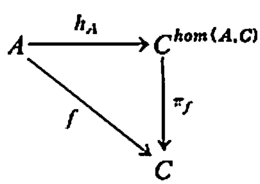
(2) *G preserves limits.*

Proof: Clearly (1) implies (2) (27.7). To show that (2) implies (1) it is sufficient to show that (2) implies that each B-object, B, G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects (28.9). Suppose that $\mathscr{R}$ is a class of pairwise non-isomorphic $\mathscr{A}$-objects and that for each $A \in \mathscr{R}$, a suitable morphism

$$B \xrightarrow{g_A} G(A)$$

G-generates A.

Now by the definition of product, for each $A \in \mathscr{R}$ there exists a morphism h_A such that for each $f \in hom(A, C)$, the diagram



commutes. Since C is a coseparator, each h_A must be a monomorphism (19.6). Since each g_A G-generates A, the function from $hom(A, C)$ to $hom(B, G(C))$ defined by $f \mapsto G(f) \circ g_A$ is injective. Thus for those objects $A \in \mathscr{R}$ for which $hom(A, C) \neq \emptyset$, there is a section $s_A: C^{hom(A,C)} \to C^{hom(B,G(C))}$ (18F), and $(A, s_A \circ h_A)$ is thus a subobject of $C^{(B,G(C))}$. If $hom(A, C) = \emptyset$, then $C^{hom(A,C)}$ is the terminal object T. Thus for each $A \in \mathscr{R}$, either $(A, s_A \circ h_A)$ is a subobject of $C^{hom(B,G(C))}$ or (A, h_A) is a subobject of T. Since $\mathscr{A}$ is well-powered, this means that $\mathscr{R}$ must be a set. □

Observe that the above theorem implies, for example, that if $\mathscr{A}$ is any one of the categories **Set**, R-**Mod**, **Top**, or **CompT$_2$**, then any functor $F: \mathscr{A} \to \mathscr{B}$ has a left adjoint if and only if it preserves limits and has a right adjoint if and only if it preserves colimits.

The following adjoint functor theorem is often applicable when the functors U and V are forgetful functors from concrete categories to the category Set.

28.12 THEOREM

If the diagram of categories and functors

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{G} & \mathscr{B} \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle V} & \\ & \mathscr{C} & \end{array}$$

commutes, and if the following conditions are satisfied:

(1) *$\mathscr{A}$ is complete, well-powered, and co-(well-powered),*
(2) *G preserves limits,*
(3) *U has a left-adjoint,*
(4) *V is faithful;*

then G has a left adjoint.

Proof: By Theorem 28.9, it suffices to show that each $\mathscr{B}$-object G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects. Let $B \in Ob(\mathscr{B})$ and suppose that $\mathscr{R}$ is a class of pairwise non-isomorphic $\mathscr{A}$-objects such that for each $A \in \mathscr{R}$, $B \xrightarrow{g_A} G(A)$ G-generates A. Since U has a left adjoint, there is some U-universal map $(u, \bar{A})$ for $V(B)$. Now for each morphism g_A,

$$V(g_A)\colon V(B) \to V(G(A)) = U(A),$$

so that since $(u, \bar{A})$ is a U-universal map, there exists a unique $\bar{g}_A\colon \bar{A} \to A$ such that the triangle

$$\begin{array}{ccccc} V(B) & \xrightarrow{u} & U(\bar{A}) & \qquad & \bar{A} \\ & {\scriptstyle V(g_A)}\searrow & \Big\downarrow {\scriptstyle U(\bar{g}_A)} & & \Big\downarrow {\scriptstyle \bar{g}_A} \\ & & U(A) = (V \circ G)(A) & & A \end{array}$$

commutes.

We claim that $\bar{g}_A$ is an epimorphism. To see this, suppose that

$$r \circ \bar{g}_A = s \circ \bar{g}_A.$$

Then

$$U(r) \circ U(\bar{g}_A) = U(s) \circ U(\bar{g}_A);$$

hence

$$(V \circ G)(r) \circ V(g_A) = (V \circ G)(s) \circ V(g_A).$$

Since V is faithful, we have $G(r) \circ g_A = G(s) \circ g_A$. But since g_A G-generates A, this implies that $r = s$. Hence each $\bar{g}_A$ is an epimorphism. Since $\mathscr{A}$ is co-(well-powered), this implies that $\mathscr{R}$ is a set. $\square$

Notice that condition (2) in the above theorem can be replaced by the condition that V reflects limits. This then eliminates all conditions on G whatsoever, except that the diagram of functors commutes.

EXERCISES

28A. Prove that if $\mathscr{A}$ is complete and $G: \mathscr{A} \to \mathscr{B}$ is a full functor which is also a surjection on objects, then the following are equivalent:

(a) G has a left adjoint.
(b) G preserves limits.

28B. For the examples of 27.5 in which $\mathscr{A}$ is complete, use the various adjoint functor theorems of this section to obtain proofs of the existence of the given adjoint situations.

28C. Let $\mathscr{C}$ be the partially-ordered class of ordinal numbers, considered as a category; let $\mathscr{B}$ be the category **1**, with only one morphism; and let $G: \mathscr{C}^{op} \to \mathscr{B}$ be the unique functor from $\mathscr{C}^{op}$ to $\mathscr{B}$.
Show that:

(a) $\mathscr{C}^{op}$ is complete.
(b) $\mathscr{C}^{op}$ is co-(well-powered).
(c) Each $\mathscr{B}$-object extremally G-generates at most a set (actually the empty set) of $\mathscr{C}^{op}$-objects.
(d) $\mathscr{C}^{op}$ has a coseparator.
(e) G preserves limits.
(f) G has no left adjoint.

Conclude that the "well-powered" hypothesis is necessary in the Second Adjoint Functor Theorem and in the Special Adjoint Functor Theorem.

28D. Let $\mathscr{A}$ be the subcategory of the category **POS** of partially ordered sets, defined as follows:
$A \in Ob(\mathscr{A})$ if and only if A has a greatest member m_A and for each $a \in A$, the order relation on A induces a well-order on the closed interval

$$[a, m_A] = \{x \mid a \leq x \leq m_A\}.$$

A morphism $f: A \to B$ in **POS** between two $\mathscr{A}$-objects is in $Mor(\mathscr{A})$ if and only if for each $a \in A$, $f[[a, m_A]] = [f(a), m_B]$ and the restriction

$$f_a: [a, inf\{x \mid a \leq x, f(x) = m_B\}] \to [f(a), m_B]$$

is a bijection.

Let $U: \mathscr{A} \to$ **Set** be the forgetful functor. Show that:

(a) $\mathscr{A}$ is complete. [For products, partially order the underlying set P of the product of the underlying sets by: $(a_i) \leq (b_i)$ if and only if $a_i \leq_i b_i$ for each i and for any two indices i and k

$$Ord[a_i, b_i] = Ord[a_k, b_k]$$

or

$$Ord[a_i, b_i] < Ord[a_k, b_k] \text{ and } b_i = m_i,$$

or

$$Ord[a_i, b_i] > Ord[a_k, b_k] \text{ and } b_k = m_k$$

(where $Ord\ X$ is the ordinal number of X).]

(b) U preserves limits.
(c) $\mathscr{A}$ is well-powered and strongly co-(well-powered).
(d) U has no left adjoint.

Conclude that the condition that each $\mathscr{B}$-object extremally G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects, in the Second Adjoint Functor Theorem, cannot be deleted.

28E. Prove the following slightly modified form of the Second Adjoint Functor Theorem:

Let $\mathscr{A}$ be complete, well-powered, and co-(well-powered). Then $G: \mathscr{A} \to \mathscr{B}$ has a left adjoint if and only if the following two conditions are satisfied:

(α) G preserves limits.
(β) Each $\mathscr{B}$-object G-generates at most a set of pairwise non-isomorphic $\mathscr{A}$-objects.

28F. Let $\mathscr{A}$ be complete and well-powered, and for each category I, let $C_I: \mathscr{A} \to \mathscr{A}^I$ be the "constant functor" functor (15.8).
(a) Show that each C_I preserves limits.
(b) Using the form of the Second Adjoint Functor Theorem proved above (28E), obtain a new proof of the fact that $\mathscr{A}$ is cocomplete if it is strongly co-(well-powered) (cf. 23.13).

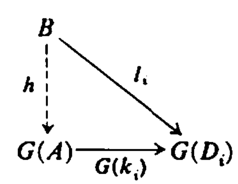
28G. Let $\mathscr{A}$ be the category of discrete topological spaces, let $\mathscr{B}$ = **Top**, and let $G: \mathscr{A} \to \mathscr{B}$ be the inclusion functor.

(a) Show that G has no left adjoint.
(b) Conclude that the condition that G preserves limits cannot be deleted from the hypothesis of Theorem 28.12.

28H. (Freyd) *An Adjoint Functor Theorem With Weak Completeness Condition*

Suppose that $\mathscr{A}$ is a category in which idempotents split (17O) and $G: \mathscr{A} \to \mathscr{B}$ is a functor. Prove that G has a left adjoint if and only if the following two conditions are satisfied:

(α) Each $\mathscr{B}$-object has a G-solution set.
(β) For each small category I and each $D: I \to \mathscr{A}$ such that $(B, (l_i))_I$ is a natural source for $G \circ D$, there exists a natural source $(A, (k_i)_I)$ for D and a morphism $h: B \to G(A)$ such that for each $i \in I$, the triangle



commutes.

[To show that (α) and (β) are sufficient, construct a universal map for each $\mathscr{B}$-object B. To do this begin with a solution set for B, considered as a functor, and use (β) to obtain a singleton solution set $(B \xrightarrow{f} G(A), A)$. Then take all $\mathscr{A}$-morphisms $h_k: A \to A$ for which $G(h_k) \circ f = f$. Consider this as a functor and use (β) again to obtain a factorization

$$B \xrightarrow{f} G(A) = B \xrightarrow{g} G(\bar{A}) \xrightarrow{G(m)} G(A).$$

Then obtain a morphism $u: A \to \bar{A}$ such that $m \circ u$ is an idempotent in $\mathscr{A}$. If

$$A \xrightarrow{m \circ u} \bar{A} = A \xrightarrow{r} \bar{A} \xrightarrow{s} \bar{A}$$

is a (retraction, section) factorization of $m \circ u$, then $(G(r) \circ f, \bar{A})$ will be a universal map for B.]