VI

Limits in Categories

Old theorems never die; they turn into definitions.

——E. HEWITT

Category theory is essentially involved with two concepts—general constructions and smallness conditions. In this chapter we consider some general constructions —namely limits and colimits—and begin to investigate the role that smallness conditions play in the relationship between them. In Chapter I we have seen that the notion of cartesian products in Set is essentially the same categorically as the notion of direct products in **Grp** or topological products in **Top**. The obvious usefulness of these concepts within their respective categories naturally leads to the categorical concept of "product". This is one particular type of a categorical limit. Other "general constructions" in well-known categories naturally lead to other varieties of limits (and colimits)—such as equalizers, kernels, intersections, inverse limits, and direct limits. Later we shall see that the knowledge of which limits and colimits exist in a given category tells much about that category, and knowledge of which limits or colimits are preserved or reflected by a given functor tells much about that functor.

§16 EQUALIZERS AND COEQUALIZERS

Equalizers

16.1 MOTIVATING PROPOSITION

If $A \underset{g}{\overset{f}{\rightrightarrows}} B$ *is a pair of functions from the set A to the set B, then the embedding e of the set*

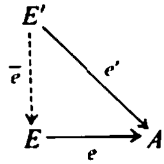
$$E = \{a \in A \mid f(a) = g(a)\}$$

into A has the following properties:

(1) *$e: E \to A$ is a function;*

(2) $f \circ e = g \circ e$;

(3) *For any function $e': E' \to A$ such that $f \circ e' = g \circ e'$, there exists a* unique *function $\bar{e}: E' \to E$ such that the triangle*



commutes. □

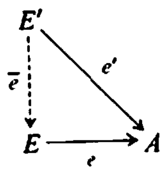
16.2 DEFINITION

Let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ be a pair of $\mathscr{C}$-morphisms. A pair (E, e) is called an **equalizer in $\mathscr{C}$ of f and g** provided that the following conditions hold:

(1) $e: E \to A$ is a $\mathscr{C}$-morphism;

(2) $f \circ e = g \circ e$;

(3) For any $\mathscr{C}$-morphism $e': E' \to A$ such that $f \circ e' = g \circ e'$, there exists a *unique* $\mathscr{C}$-morphism $\bar{e}: E' \to E$ such that the triangle



commutes.

DUALLY: If $c: B \to C$, then (c, C) is called a **coequalizer in $\mathscr{C}$** of a pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ if and only if $c \circ f = c \circ g$ and each morphism c' with the property that $c' \circ f = c' \circ g$ can be uniquely factored through c.

16.3 EXAMPLES

(1) If $\mathscr{C}$ is one of the categories **Set**, **Grp**, *R*-**Mod**, or **Top** and if $A \overset{f}{\underset{g}{\rightrightarrows}} B$ are $\mathscr{C}$-morphisms, then if E denotes the set $\{a \in A \mid f(a) = g(a)\}$ considered as a subset (resp. subgroup, submodule, subspace) of A and if $e: E \to A$ is the inclusion map, then (E, e) is an equalizer of f and g.

(2) If $\mathscr{C}$ is one of the categories **Set** or **Top** (resp. **Grp** or *R*-**Mod**) and if $A \overset{f}{\underset{g}{\rightrightarrows}} B$ are $\mathscr{C}$-morphisms, let Q be the smallest equivalence relation (resp. congruence) on B that contains all pairs $(f(a), g(a))$ for $a \in A$, let C be B/Q with the

appropriate induced structure, and let $c: B \to C$ be the induced quotient map. Then (c, C) is a coequalizer of f and g.

(3) Let $\mathscr{C}$ be the category of locally connected spaces and continuous functions. If $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are $\mathscr{C}$-morphisms, and $E = \{x \mid f(x) = g(x)\}$ is supplied with the coarsest locally connected topology for which the inclusion $e: E \to X$ is continuous, then (E, e) is an equalizer in $\mathscr{C}$ of f and g.

16.4 PROPOSITION

If (E, e) is an equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$, then (E, e) is a subobject of A.

Proof: We must show that e is a monomorphism (6.22). Suppose that $e \circ r = e \circ s$. Then $f \circ (e \circ r) = g \circ (e \circ r)$ so that by the definition of equalizer, there is a unique morphism t such that $e \circ t = e \circ r$. But each of s and r is such a morphism; hence $s = r$. Thus e is a monomorphism. □

16.5 PROPOSITION (UNIQUENESS OF EQUALIZERS)

Any two equalizers of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ are isomorphic subobjects of A.

Proof: If each of (E, e) and $(\hat{E}, \hat{e})$ is an equalizer of f and g, then by the definition of equalizer there exist unique morphisms p and q such that $e = \hat{e} \circ p$ and $\hat{e} = e \circ q$. Thus $(E, e) \leq (\hat{E}, \hat{e})$ and $(\hat{E}, \hat{e}) \leq (E, e)$; i.e., (E, e) and $(\hat{E}, \hat{e})$ are isomorphic subobjects of A (6.23). □

16.6 The above proposition shows that there is no essential difference between two equalizers of a pair of morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$. Because of this, we often speak (loosely) of *the* equalizer of f and g [denoted by $Equ(f, g)$]. The notation $(E, e) \approx Equ(f, g)$ will be used to mean that the subobject (E, e) of A is an equalizer of f and g. [Sometimes this is abbreviated (inaccurately) to $e \approx Equ(f, g)$.]

DUALLY: $Coeq(f, g)$; $(c, C) \approx Coeq(f, g)$ or $c \approx Coeq(f, g)$.

16.7 PROPOSITION

If $(E, e) \approx Equ(f, g)$, then the following are equivalent:

(1) $f = g$.

(2) *e is an isomorphism.*

(3) *e is an epimorphism.*

Proof:

(1) ⇒ *(2)*. If $f = g$, then $f \circ 1 = g \circ 1$ so that there is a morphism s such that $e \circ s = 1$. Hence e is a retraction and (since (E, e) is a subobject) a monomorphism; hence an isomorphism (6.7).

(2) ⇒ *(3)*. Trivial.

(3) ⇒ *(1)*. Since e is an epimorphism, $f \circ e = g \circ e$ implies that $f = g$. □

Up to this point nothing has been said about the *existence* of equalizers of pairs of morphisms with common domain and common codomain. It should be stressed that one cannot, in general, assume that they exist within a given category.

16.8 DEFINITION

A category $\mathscr{C}$ **has equalizers** provided that every pair of $\mathscr{C}$-morphisms with common domain and common codomain has an equalizer.

DUAL NOTION: **has coequalizers.**

16.9 EXAMPLES

(1) Each of the categories **Set**, **Grp**, R-**Mod**, and **Top** has equalizers and coequalizers.

(2) The category of all sets, with at least two members and functions between them, has neither equalizers nor coequalizers.

16.10 DEFINITION

(1) If $(h_i)_{i \in I}$ is a non-empty indexed family of morphisms contained in $hom_{\mathscr{C}}(A, B)$, then (E, e) is said to be a **multiple equalizer of** $(h_i)_I$, denoted by $(E, e) \approx Equ((h_i)_I)$, provided that:

(i) $e: E \to A$;

(ii) For all $i, j \in I$, $h_i \circ e = h_j \circ e$;

(iii) If $e': E' \to A$ such that $h_i \circ e' = h_j \circ e'$ for all $i, j \in I$, then there is a unique morphism $\bar{e}$ such that $e \circ \bar{e} = e'$.

(2) A category $\mathscr{C}$ **has multiple equalizers** provided that each non-empty indexed family of morphisms that have a common domain and a common codomain, has a multiple equalizer.

DUAL NOTIONS: **multiple coequalizer; has multiple coequalizers.**

16.11 PROPOSITION

Each multiple equalizer is a subobject. □

16.12 PROPOSITION (UNIQUENESS OF MULTIPLE EQUALIZERS)

Any two multiple equalizers of the same family of morphisms are isomorphic subobjects. □

Regular Monomorphisms

16.13 DEFINITION

(1) If $E \xrightarrow{e} A$ is a $\mathscr{C}$-morphism, then (E, e) is called a **regular subobject**† of A and e is called a **regular monomorphism**†† if and only if there are $\mathscr{C}$-morphisms f and g such that $(E, e) \approx Equ(f, g)$.

† Sometimes only the object E is (inaccurately) called a regular subobject of A if there is some e such that (E, e) is a regular subobject of A.

†† The reason for not calling these special morphisms "equalizers" lies in the fact that this would lead to undue confusion when we define what it means for a functor to preserve equalizers. It will turn out that a functor may preserve regular monomorphisms without preserving equalizers (see Exercise 24B).

(2) A category is called **regular well-powered** provided that each $\mathscr{C}$-object has a representative set of regular subobjects.

DUAL NOTIONS: **regular quotient object**; **regular epimorphism**; **regular co-(well-powered).**

16.14 EXAMPLES

(1) In **Set**, **Grp**, R-**Mod**, and **CompT$_2$** the regular monomorphisms are precisely the monomorphisms and the regular epimorphisms are precisely the epimorphisms. [To see that monomorphisms in **Grp** are regular, see Exercise 6H(a).]

(2) In **Top** the regular monomorphisms are the embeddings (i.e., homeomorphisms into) and the regular epimorphisms are the topological quotient maps (i.e., surjective identification maps).

(3) In **Top$_2$** the regular monomorphisms are precisely the closed embeddings. [If $X \xrightarrow{f} Y$ is a closed embedding (i.e., a homeomorphism onto a closed subset), then let $Y_1 = Y_2 = Y$, let $Y_1 \amalg Y_2$ denote the disjoint topological union of Y_1 and Y_2, let

$$\mu_i: Y_i \to Y_1 \amalg Y_2$$

for $i = 1, 2$ be the corresponding embeddings, and let

$$q: Y_1 \amalg Y_2 \to Z$$

be the quotient map that identifies for each $x \in X$ the two points $\mu_1(f(x))$ and $\mu_2(f(x))$. Then Z is Hausdorff and $(X, f) \approx Equ(q \circ \mu_1, q \circ \mu_2)$.]

(4) In **Mon**, **SGrp**, and **Rng** there are monomorphisms that are not regular monomorphisms; e.g., the inclusion $\mathbf{Z} \hookrightarrow \mathbf{Q}$.

16.15 PROPOSITION

In any category $\mathscr{C}$:

(1) *every $\mathscr{C}$-section is a regular monomorphism in $\mathscr{C}$; and*

(2) *every regular monomorphism in $\mathscr{C}$ is a $\mathscr{C}$-monomorphism.*

Proof: (2) is immediate from Proposition 16.4. To show (1), assume that $A \xrightarrow{f} B$ is a section. Then there is a morphism $B \xrightarrow{g} A$ such that $g \circ f = 1_A$. We claim that $(A, f) \approx Equ(1_B, f \circ g)$. Clearly $1_B \circ f = f \circ 1_A = f \circ g \circ f$. Also if r is a morphism such that $1_B \circ r = (f \circ g) \circ r$, then $r = f \circ (g \circ r)$. Thus r factors through f. The factorization is unique since f (being a section) is a monomorphism (6.6). Hence, f is a regular monomorphism. □

In general, regular monomorphisms lie strictly between sections and monomorphisms since, for example, in **Top** any bijective function from a discrete space to a non-discrete space is a monomorphism that is not regular, and the embedding of the unit circle into the unit disc is a regular monomorphism that is not a section. In general, regular monomorphisms fail to satisfy certain of the convenient conditions whose analogues have already been established for monomorphisms and sections; for example (even in the "respectable" category **SGrp**) the composition of regular monomorphisms is not necessarily a regular monomorphism and if $f = g \circ h$ is a regular monomorphism, h is not necessarily a regular monomorphism (see Exercises 16J and 34K). Nonetheless, we do have the following convenient characterization of isomorphisms in terms of regular morphisms.

16.16 PROPOSITION

For any morphism f, the following are equivalent:

(1) *f is an isomorphism.*

(2) *f is a regular monomorphism and an epimorphism.*

(3) *f is a regular epimorphism and a monomorphism.*

Proof: Since every isomorphism is a section, and each section is a regular monomorphism, (1) implies (2). That (2) implies (1) is an immediate consequence of Proposition 16.7. Clearly (3) is the dual of (2), and (1) is self-dual. □

Kernels

An important special case of equalizers and coequalizers in **Grp** and R-**Mod** has been observed quite early—the concepts of kernels and cokernels.

16.17 DEFINITION

Let $\mathscr{C}$ be a pointed category.

(1) If $A \xrightarrow{f} B$ is a $\mathscr{C}$-morphism and if 0_{AB} is the unique zero morphism from A to B, then (if it exists) $Equ(f, 0_{AB})$ is called **the kernel of** f. *Notation:* $Ker(f)$.

(2) $\mathscr{C}$ is said to **have kernels** provided that $Ker(f)$ exists for each $f \in Mor(\mathscr{C})$.

(3) If $K \xrightarrow{k} A$ is a $\mathscr{C}$-morphism, then (K, k) is called a **normal subobject of** A, and k is called a **normal monomorphism** provided that there is some morphism f in $\mathscr{C}$ such that

$$(K, k) \approx Ker(f).$$

DUAL NOTIONS: **the cokernel of** f; $Cok(f)$; **has cokernels; normal quotient object; normal epimorphism.**

16.18 EXAMPLES

(1) If $A \xrightarrow{f} B$ is a morphism in **Ab**, R-**Mod**, **pSet**, or **pTop**, and if 0 is the distinguished element of B, then

$$(f^{-1}[\{0\}], i) \approx Ker(f) \quad \text{and} \quad (\natural, B/f[A]) \approx Cok(f),$$

where i is the inclusion and $\natural$ is the natural map from B to the quotient. Thus, these categories have kernels and cokernels.

(2) The normal monomorphisms in **Grp** are (up to isomorphism) the embeddings of normal subgroups. Hence in **Grp** a regular monomorphism need not be

normal. Also in **Grp**, the composition of normal monomorphisms is not necessarily normal. [If A_4 is the alternating group on the four elements 1, 2, 3, and 4, then

$$V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$$

is a normal subgroup of A_4 and

$$Z_2 = \{(1), (12)(34)\}$$

is a normal subgroup of V_4, yet Z_2 is not normal in A_4.]

EXERCISES

16A. If $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are $\mathscr{C}$-morphisms, show that the following are equivalent:
(a) $f = g$.
(b) $(A, 1_A) \approx Equ(f, g)$.
(c) For each isomorphism $C \xrightarrow{h} A$, $(C, h) \approx Equ(f, g)$.

16B. Suppose that m is a monomorphism and that $m \circ f$ and $m \circ g$ are defined. Show that the following are equivalent:
(a) $(E, e) \approx Equ(f, g)$.
(b) $(E, e) \approx Equ(m \circ f, m \circ g)$.

16C. Show that if $\mathscr{C}$ has equalizers and I is the category $\overset{1}{\bullet} \underset{n}{\overset{m}{\rightrightarrows}} \overset{2}{\bullet}$, then there exists a functor $F: \mathscr{C}^I \to \mathscr{C}$ and a natural transformation $e = (e_D)$ from F to the evaluation functor relative to 1, $E(\_\_, 1): \mathscr{C}^I \to \mathscr{C}$ such that for each $D: I \to \mathscr{C}$ the pair $(F(D), e_D)$ is an equalizer of $D(m)$ and $D(n)$.

16D. Show that $\mathbf{BanSp_1}$ and $\mathbf{BanSp_2}$ have equalizers.

16E. Show that the category of torsion free abelian groups has coequalizers and that they are formed by factoring out the torsion subgroup after forming the usual coequalizer in **Ab**.

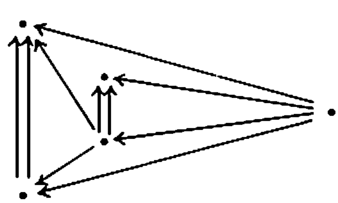
16F. Show that any non-trivial group (considered as a category) does not have equalizers.

16G. Suppose that $\mathscr{C}$ has equalizers. Prove that f is a $\mathscr{C}$-epimorphism if and only if whenever $f = m \circ g$ and m is a regular monomorphism, then m is an isomorphism.

16H. Determine the regular monomorphisms in **POS**.

16I. Prove that if $g \circ f$ is a regular monomorphism and f is an epimorphism, then f is an isomorphism.

16J. (a) Show that



is a category and conclude that the composition of regular monomorphisms is not necessarily a regular monomorphism.
(b) Construct a category to show that if $g \circ h$ is a regular monomorphism, then h is not necessarily a regular monomorphism.

16K. Prove that if f is a morphism in R-**Mod**, then the following are equivalent:
(a) f is an injective function on the underlying sets.
(b) f is a monomorphism.
(c) f is a regular monomorphism.
(d) f is a normal monomorphism.

16L. *Kernels in Pointed Categories*
(a) Show that if $K \xrightarrow{k} X$ is a morphism and $X \xrightarrow{f} Y$ is a monomorphism in a pointed category, then the following are equivalent:
(i) $(K, k) \approx Ker(f)$.
(ii) K is a zero object.
(b) Show that each non-empty pointed category that has kernels has a zero object.
(c) Show that the following conditions are equivalent for the categories R-**Mod** and **Grp**, but not for **Mon** or **pSet**.
(i) $f: X \to Y$ is a monomorphism.
(ii) $Ker(f) \approx (0, 0_{0X})$
(where 0 is the zero object and 0_{0X} is the zero morphism from 0 to X).
(d) Show that for each morphism f in a category that has a zero object 0, $Ker(Ker(f)) \approx 0$.

16M. If $\mathscr{C}$ is pointed and has kernels, show that there is a functor $K: \mathscr{C}^2 \to \mathscr{C}$ such that for each $f \in Ob(\mathscr{C}^2)$, there is a morphism k, such that $(K(f), k) \approx Ker(f)$.

16N. Let $(\mathscr{C}, U)$ be a concrete category. Prove that
(a) $\mathscr{C}$ is regular co-(well-powered). [Associate with each regular epimorphism $e: A \to B$ the equivalence relation

$$\{(a, b) \mid a, b \in U(A) \quad \text{and} \quad U(e)(a) = U(e)(b)\}.]$$

(b) $\mathscr{C}$ is regular well-powered.

16O. (a) Prove that the forgetful functor from any of the categories **Grp**, R-**Mod**, **SGrp**, **Mon**, **Lat**, **BooAlg**, or **CompT$_2$** into **Set** preserves and reflects regular epimorphisms.
(b) Prove that the forgetful functor from any of the categories **Top**, **Top$_2$**, or **POS** into **Set** preserves but does not reflect regular epimorphisms.
(c) Let U: **Cat** $\to$ **Set** be the forgetful functor that associates with each small category its set of morphisms. Prove that U neither preserves nor reflects regular epimorphisms (cf. Exercise 11C).

§17 INTERSECTIONS AND FACTORIZATIONS

Intersections

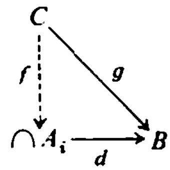
17.1 MOTIVATING PROPOSITION

If for each $i \in I$, A_i is a subset of the set B and m_i: $A_i \hookrightarrow B$ is the inclusion, then the intersection $\cap A_i$ and its inclusion d into B have the following properties:

(1) *$d\colon \cap A_i \to B$ is a function;*

(2) *for each $i \in I$ there is a function $d_i\colon \cap A_i \to A_i$ with the property that $m_i \circ d_i = d$;*

(3) *if $g\colon C \to B$ and for each $i \in I$, $g_i\colon C \to A_i$ such that $m_i \circ g_i = g$, then there exists a unique morphism $f\colon C \to \cap A_i$ such that the triangle*



commutes. □

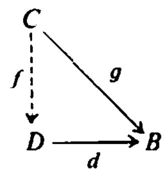
17.2 DEFINITION

If B is a $\mathscr{C}$-object and $(A_i, m_i)_I$ is a family of subobjects of B, then the pair (D, d) is called an **intersection in $\mathscr{C}$ of $(A_i, m_i)_I$** provided that

(1) $d\colon D \to B$ is a $\mathscr{C}$-morphism;

(2) for each $i \in I$ there is a $\mathscr{C}$-morphism $d_i\colon D \to A_i$ with the property that $m_i \circ d_i = d$;

(3) if $g\colon C \to B$ and for each $i \in I$, $g_i\colon C \to A_i$ such that $m_i \circ g_i = g$, then there exists a unique $\mathscr{C}$-morphism $f\colon C \to D$ such that the triangle



commutes.

DUAL NOTION: **Cointersection** of a family of quotient objects.

17.3 PROPOSITION

Every intersection (D, d) of a family of subobjects $(A_i, m_i)_I$ of an object B is itself a subobject of B; i.e., d is necessarily a monomorphism. Furthermore (D, d) is (up to isomorphism) the largest subobject (relative to the order $\leq$ on subobjects) that is smaller than each of the subobjects (A_i, m_i).

Proof: That d is a monomorphism follows from the uniqueness condition in the definition of intersection (cf. 16.4). The last statement in the proposition is immediate from the definitions of intersection and order on subobjects (6.23). □

For more about the relationship between intersections and the order relation on subobjects, see Exercises 17A and 34E.

17.4 COROLLARY

Any two intersections of a family of subobjects of an object A are isomorphic subobjects of A. □

Since, by the above corollary, any two intersections of a family of subobjects of some object are essentially the same, we often speak (loosely) of *the* intersection of such a family.

17.5 DEFINITION

A category $\mathscr{C}$ **has intersections** (resp. **has finite intersections**) provided that every set-indexed family (resp. finite family) of subobjects of each $\mathscr{C}$-object has an intersection.

17.6 EXAMPLES

(1) For any object B, the empty family of subobjects of B has an intersection, namely $(B, 1_B)$.
(2) If for each $i \in I$, A_i is a subset of the set B and $m_i\colon A_i \hookrightarrow B$ is the inclusion, and if $d\colon \cap A_i \hookrightarrow B$ is the usual inclusion, then $(\cap A_i, d)$ is the intersection in **Set** of the family $(A_i, m_i)_I$ of subobjects of B.
(3) In a manner similar to that used for sets in (2), the "usual" construction of the intersection of a family of subgroups of a given group or of the intersection of a family of subspaces of a given topological space can be shown to correspond to the "categorical" intersections in **Grp** and **Top** respectively.

17.7 PROPOSITION

If $\mathscr{C}$ is well-powered and has intersections, then every (not necessarily set-indexed) family of subobjects of any $\mathscr{C}$-object has an intersection.

Proof: Let $(A_i, m_i)_I$ be a family of subobjects of B. Since $\mathscr{C}$ is well-powered, there is a representative set-indexed subfamily of $(A_i, m_i)_I$. Clearly the intersection of the set-indexed subfamily is also the intersection of the original family. □

Factorizations and Extremal Morphisms

17.8 PROPOSITION

If $\mathscr{C}$ is well-powered and has intersections and equalizers, then every $\mathscr{C}$-morphism $A \xrightarrow{f} B$ has a factorization

$$A \xrightarrow{f} B = A \xrightarrow{e} C \xrightarrow{m} B$$

where e is an epimorphism and m is a monomorphism.

Proof: Let $(D_i, m_i)_I$ be the family of all subobjects of B which are part of some factorization

$$A \xrightarrow{f} B = A \xrightarrow{h_i} D_i \xrightarrow{m_i} B$$

of f. By the above proposition (17.7), $(D_i, m_i)_I$ has an intersection (D, m), which is itself a subobject (17.3). Now by the definition of intersection, there exists a unique morphism $e\colon A \to D$ such that $f = m \circ e$.

To show that e is an epimorphism, let $r \circ e = s \circ e$, and let $(E, k) \approx Equ(r, s)$. By the definition of equalizer (16.2) there is a morphism $h\colon A \to E$

such that $e = k \circ h$. Since k is a monomorphism (16.4), and since monomorphisms are closed under composition, $(E, m \circ k)$ belongs to the family $(D_i, m_i)_I$; i.e., for some $j \in I$, $(E, m \circ k) = (D_j, m_j)$. Thus,

$$m \circ k \circ d_j = m_j \circ d_j = m = m \circ 1$$

so that since m is a monomorphism, $k \circ d_j = 1$; i.e., k is a retraction (and a monomorphism). Consequently, k is an isomorphism, so that $r = s$ (16.7). □

It often occurs that a morphism f can be written as a composition of an epimorphism e followed by a monomorphism m in several essentially different ways (cf. Exercise 17E). The factorization $f = m \circ e$ constructed above is characterized by the fact that the epimorphism e has an additional important property defined below.

17.9 DEFINITION

(1) A morphism e is called an **extremal epimorphism** provided that it satisfies the following two conditions:

(i) e is an epimorphism.

(ii) (*Extremal condition*): If $e = m \circ f$, where m is a monomorphism, then m must be an isomorphism.

(2) If $A \xrightarrow{e} B$ is an extremal epimorphism, then (e, B) is called an **extremal quotient object of** A.

(3) A category is called **extremally co-(well-powered)** provided that each $\mathscr{C}$-object has at most a set of non-isomorphic extremal quotient objects.

DUAL NOTIONS: **extremal monomorphism**; **extremal subobject**; **extremally well-powered.**

17.10 EXAMPLES

(1) In the categories **Set**, **Grp**, and R-**Mod**, extremal epimorphisms are the same as epimorphisms and extremal monomorphisms are the same as monomorphisms.

(2) In the category **Top** extremal epimorphisms are topological quotient maps and extremal monomorphisms are (up to homeomorphism) embeddings.

(3) In the category $\mathbf{Top}_2$ extremal epimorphisms are topological quotient maps and extremal monomorphisms are (up to homeomorphism) closed embeddings.

(4) In **SGrp** and **Rng** extremal epimorphisms are surjective homomorphisms and there are monomorphisms that are not extremal.

(5) In $\mathbf{BanSp}_1$ a morphism $X \xrightarrow{f} Y$ is an extremal epimorphism if and only if it is a surjective bounded linear transformation and it is an extremal monomorphism provided that there is some $m > 0$ such that for all $x \in X$

$$m\|x\| \leq \|f(x)\|.$$

17.11 PROPOSITION

Every regular epimorphism is an extremal epimorphism.

Proof: Clearly every regular epimorphism is an epimorphism (16.15 dual). Now suppose that $A \xrightarrow{e} B$ is a regular epimorphism and $e = m \circ f$, where m is a monomorphism. We know that there exist morphisms r and s such that $(e, B) \approx Coeq(r, s)$. Thus

$$m \circ f \circ r = e \circ r = e \circ s = m \circ f \circ s$$

so that since m is a monomorphism, $f \circ r = f \circ s$. Hence by the definition of coequalizer, there exists a morphism h such that $h \circ e = f$. Thus

$$m \circ h \circ e = m \circ f = e = 1 \circ e$$

so that since e is an epimorphism, $m \circ h = 1$. Thus m is a retraction and (by hypothesis) a monomorphism. Consequently, m is an isomorphism. □

17.12 PROPOSITION

For any morphism f, the following are equivalent:

(1) *f is an isomorphism.*

(2) *f is an extremal epimorphism and a monomorphism.*

(3) *f is an extremal monomorphism and an epimorphism.*

Proof:

(1) ⇒ *(2)*. This follows immediately from the facts that each isomorphism is a regular epimorphism and a monomorphism (16.16) and that each regular epimorphism is an extremal epimorphism (17.11).

(2) ⇒ *(1)*. Suppose that f is an extremal epimorphism and a monomorphism. Then $f = f \circ 1$, where f is a monomorphism, so that by the extremal condition, f must be an isomorphism.

(1) ⇔ *(3)*. This is immediate from the fact that (1) is equivalent to (2), (1) is self-dual and (3) is the dual of (2). □

17.13 PROPOSITION

For any category $\mathscr{C}$, the following are equivalent:

(1) *$\mathscr{C}$ is balanced.*

(2) *Each $\mathscr{C}$-epimorphism is an extremal epimorphism.*

(3) *Each $\mathscr{C}$-monomorphism is an extremal monomorphism.*

Proof:

(1) ⇒ *(2)*. If e is an epimorphism and $e = m \circ f$, where m is a monomorphism, then m is also an epimorphism (6.13), so that since $\mathscr{C}$ is balanced, m is an isomorphism.

(2) ⇒ *(1)*. Immediate from the preceding proposition (17.12).

(1) ⇔ *(3)*. Immediate from the fact that (1) is equivalent to (2), (1) is self-dual and (3) is the dual of (2). □

17.14 PROPOSITION

If $\mathscr{C}$ has equalizers and e is a $\mathscr{C}$-morphism that satisfies the extremal condition (ii) *of Definition* 17.9(1), *then e must be an extremal epimorphism.*

Proof: We need only show that e is an epimorphism. Suppose that r and s are $\mathscr{C}$-morphisms such that $r \circ e = s \circ e$. Let $(K, k) \approx Equ(r, s)$. Then k is a monomorphism (16.4) and by the definition of equalizer there is a morphism h such that $e = k \circ h$. Hence, since e satisfies the extremal condition, k must be an isomorphism. Thus $r = s$ (16.7). □

17.15 DEFINITION

Let $\mathscr{E}$ and $\mathscr{M}$ be classes of morphisms of a category $\mathscr{C}$.

(1) A pair (e, m) is called an **$(\mathscr{E}, \mathscr{M})$-factorization** of a $\mathscr{C}$-morphism f provided that:

(i) $f = m \circ e$

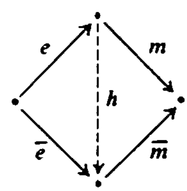
(ii) $e \in \mathscr{E}$

(iii) $m \in \mathscr{M}$

This is sometimes abbreviated by saying that $f = m \circ e$ is an $(\mathscr{E}, \mathscr{M})$-factorization of f.

(2) $\mathscr{C}$ is called an **$(\mathscr{E}, \mathscr{M})$-factorizable category** provided that each $\mathscr{C}$-morphism has an $(\mathscr{E}, \mathscr{M})$-factorization.

(3) $\mathscr{C}$ is called a **uniquely $(\mathscr{E}, \mathscr{M})$-factorizable category** if and only if it is $(\mathscr{E}, \mathscr{M})$-factorizable and for any two $(\mathscr{E}, \mathscr{M})$-factorizations $f = m \circ e = \bar{m} \circ \bar{e}$ of the same $\mathscr{C}$-morphism f, there exists an isomorphism h such that the diagram



commutes.

(4) If $\mathscr{M}$ is composed of monomorphisms, then (A, f) is called an **$\mathscr{M}$-subobject** provided that $f \in \mathscr{M}$ and $\mathscr{C}$ is called **$\mathscr{M}$-well-powered** provided that each $\mathscr{C}$-object has a representative set of $\mathscr{M}$-subobjects.

DUALLY: if $\mathscr{E}$ is composed of epimorphisms, we have the concepts: **$\mathscr{E}$-quotient object** and **$\mathscr{E}$-co-(well-powered)**.

In particular, if $\mathscr{E}$ is the class of all extremal epimorphisms (resp. regular epimorphisms) of $\mathscr{C}$ and $\mathscr{M}$ is the class of all $\mathscr{C}$-monomorphisms, an $(\mathscr{E}, \mathscr{M})$-factorization is called an **(extremal epi, mono)-factorization** [resp. **(regular epi, mono)-factorization**], and if $\mathscr{E}$ is the class of all $\mathscr{C}$-epimorphisms and $\mathscr{M}$ is the class of all extremal monomorphisms (resp. regular monomorphisms) in $\mathscr{C}$, an $(\mathscr{E}, \mathscr{M})$-factorization is called an **(epi, extremal mono)-factorization** [resp. **(epi, regular mono)-factorization**], etc.

17.16 PROPOSITION

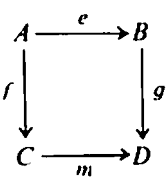
If $\mathscr{C}$ is well-powered and has intersections and equalizers, then $\mathscr{C}$ is (extremal epi, mono)-factorizable.

Proof: By Proposition 17.8 we know that each $\mathscr{C}$-morphism f has an (epi, mono)-factorization $f = m \circ e$. By examining the proof of 17.8, it is easy to verify that e also satisfies the extremal condition so that $f = m \circ e$ is in fact an (extremal epi, mono)-factorization of f. □

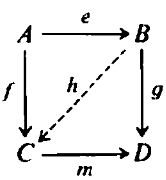
Later (§34) we will show that under the above hypotheses, $\mathscr{C}$ is even *uniquely* (extremal epi, mono)-factorizable.

17.17 PROPOSITION

If the square



commutes and if e is a regular epimorphism and m is a monomorphism, then there exists a morphism $h\colon B \to C$ such that the diagram



commutes.

Proof: Since e is a regular epimorphism, there are morphisms r and s such that $(e, B) \approx Coeq(r, s)$. Thus

$$m \circ (f \circ r) = g \circ (e \circ r) = g \circ (e \circ s) = m \circ (f \circ s),$$

so that since m is a monomorphism, $f \circ r = f \circ s$. Hence, by the definition of coequalizer, there is a morphism h such that $f = h \circ e$. Since e is an epimorphism, it also follows that $g = m \circ h$. □

17.18 PROPOSITION

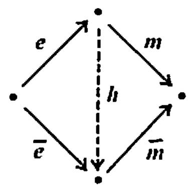
If a category $\mathscr{C}$ is (regular epi, mono)-factorizable, then

(1) *$\mathscr{C}$ is uniquely (regular epi, mono)-factorizable.*

(2) *The regular epimorphisms in $\mathscr{C}$ are precisely the extremal epimorphisms.*

Proof:

(*1*). If each of $f = m \circ e$ and $f = \overline{m} \circ \overline{e}$ is a (regular epi, mono)-factorization of f, then by proposition (17.17) there is a morphism h such that the diagram



commutes. Being the first factor of the monomorphism m, h must be a monomorphism (6.5). However, since $\overline{e}$ is a regular epimorphism, it is an extremal epimorphism (17.11). Thus by the extremal condition, h must be an isomorphism.

(*2*). Immediate. ☐

A further treatment of extremal morphisms and factorizations appears in Chapter IX.

EXERCISES

17A. Let $(X_i, m_i)_I$ be a family of subobjects of the object X and let (D, d) be a subobject of X such that
(a) for each i, $(D, d) \leq (X_i, m_i)$ and
(b) if (E, e) is a subobject of X such that for each i, $(E, e) \leq (X_i, m_i)$, then $(E, e) \leq (D, d)$.
Show by means of an example that (D, d) is not necessarily the intersection of $(X_i, m_i)_I$.

17B. Prove that if $\mathscr{C}$ is (epi, mono)-factorizable and f is a morphism that satisfies the extremal condition (ii) of Definition 17.9(1), then f is an extremal epimorphism.

17C. Prove that if f is an extremal monomorphism and $f = h \circ g$, then g must be an extremal monomorphism (cf. Propositions 5.5 and 6.5 and Exercise 16J).

17D. *Splitting of Idempotents*
An **idempotent** in a category $\mathscr{C}$ is a $\mathscr{C}$-morphism $f: A \to A$ with the property that $f \circ f = f$. An idempotent f is said to **split** provided that there is a factorization

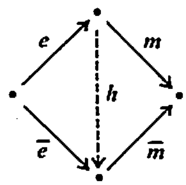
$$A \xrightarrow{f} A = A \xrightarrow{r} B \xrightarrow{s} A$$

where $r \circ s = 1_B$.

Show that the following statements are equivalent for any category $\mathscr{C}$ and any idempotent morphism $A \xrightarrow{f} A$ in $\mathscr{C}$.
(a) f splits in $\mathscr{C}$.
(b) f has a (retraction, section)-factorization in $\mathscr{C}$.
(c) The morphisms f and 1_A have an equalizer in $\mathscr{C}$.
(d) The morphisms f and 1_A have a coequalizer in $\mathscr{C}$.

17E. Show that in the categories **Top** and **Rng** there are morphisms f that have two essentially different (epi, mono)-factorizations; i.e., $f = m \circ e$ and $f = \overline{m} \circ \overline{e}$

where e and $\bar{e}$ are epimorphisms, m and $\bar{m}$ are monomorphisms, and there is no isomorphism h such that the diagram

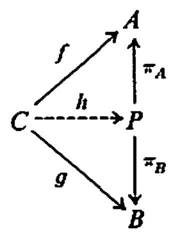


commutes.

§18 PRODUCTS AND COPRODUCTS

18.1 MOTIVATING PROPOSITION

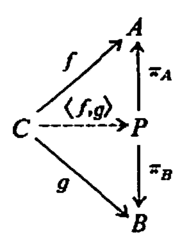
The cartesian product of a pair (A, B) of sets is a set P together with two (projection) functions $\pi_A: P \to A$ and $\pi_B: P \to B$ with the property that if C is any set and $f: C \to A$, $g: C \to B$ are functions, then there exists a unique function $h: C \to P$ such that the diagram



commutes. □

18.2 DEFINITION

A $\mathscr{C}$-**product** of a pair (A, B) of $\mathscr{C}$-objects is a triple (P, π_A, π_B) where P is a $\mathscr{C}$-object and $\pi_A: P \to A$, $\pi_B: P \to B$ are $\mathscr{C}$-morphisms (called **projections**) with the property that if C is any $\mathscr{C}$-object and $f: C \to A$, $g: C \to B$ are arbitrary $\mathscr{C}$-morphisms, then there exists a *unique* $\mathscr{C}$-morphism (usually denoted by) $\langle f, g\rangle: C \to P$ such that the diagram



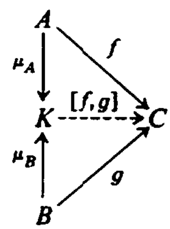
commutes.

Quite often the triple (P, π_A, π_B) is denoted by $A \times B$. Usually the symbol $A \times B$ is (inaccurately) used to stand for just the object P, rather than the entire product. It should be kept in mind, however, that a product is really a triple—not just an object.

The notion that is dual to product is coproduct, the definition of which can, of course, be readily obtained by forming the definition of product in $\mathscr{C}^{op}$ and translating this back into a statement in $\mathscr{C}$ (4.15). However because of its importance and to make the notion explicit, we state the definition.

18.3 DEFINITION

A **$\mathscr{C}$-coproduct** of a pair (A, B) of $\mathscr{C}$-objects is a triple (μ_A, μ_B, K) where K is a $\mathscr{C}$-object and $\mu_A: A \to K$, $\mu_B: B \to K$ are $\mathscr{C}$-morphisms (called **injections**) with the property that if C is any $\mathscr{C}$-object and $f: A \to C$ $g: B \to C$ are arbitrary $\mathscr{C}$-morphisms, then there exists a *unique* $\mathscr{C}$-morphism (usually denoted by) $[f, g]: K \to C$ such that the diagram



commutes.

The triple (μ_A, μ_B, K), and often just K, is usually denoted by $A \amalg B$.

18.4 EXAMPLES

	Category	$A \times B$	$A \amalg B$
(1)	**Set**	cartesian product	disjoint union
(2)	**Grp**	direct product	free product
(3)	***R*-Mod**	direct product	direct sum
(4)	**Top**	topological product	topological (disjoint) sum
(5)	**pTop**	topological product	topological sum with base points identified
(6)	**TopBun**$_B$	fibre product (Whitney sum)	
(7)	commutative R-**Alg**	direct product	tensor product $A \otimes_R B$
(8)	**Cat** or $\mathscr{CAT}$	product category (9L)	sum category (9M)
(9)	a single partially ordered set (3.5(6))	infimum $= A \wedge B$	supremum $= A \vee B$

It is clear that one can easily generalize the notion of products of pairs of objects to the notion of finite products. However, as in set theory, it is useful to consider products for arbitrary families of objects, to wit—

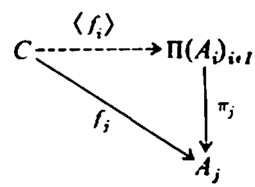
18.5 DEFINITION

A **$\mathscr{C}$-product** of a family $(A_i)_{i\in I}$ of $\mathscr{C}$-objects is a pair (usually denoted by) $(\Pi(A_i)_{i\in I}, (\pi_i)_{i\in I})$ satisfying the following properties:

(1) $\Pi(A_i)_{i\in I}$ is a $\mathscr{C}$-object.

(2) for each $j \in I$, π_j: $\Pi(A_i)_{i\in I} \to A_j$ is a $\mathscr{C}$-morphism (called the **projection from $\Pi(A_i)_{i\in I}$ to A_j**).

(3) for each pair $(C, (f_i)_{i\in I})$, (where C is a $\mathscr{C}$-object and for each $j \in I, f_j: C \to A_j$) there exists a unique $\mathscr{C}$-morphism (usually denoted by) $\langle f_i \rangle: C \to \Pi(A_i)_{i\in I}$ such that for each $j \in I$, the triangle



commutes.

18.6 DEFINITION

A **$\mathscr{C}$-coproduct** of a family $(A_i)_{i\in I}$ of $\mathscr{C}$-objects is a pair (usually denoted by) $((\mu_i)_{i\in I}, \amalg(A_i)_{i\in I})$ satisfying the following properties:

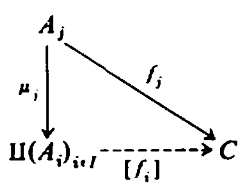
(1) $\amalg(A_i)_{i\in I}$ is a $\mathscr{C}$-object.

(2) for each $j \in I$, μ_j: $A_j \to \amalg(A_i)_{i\in I}$ is a $\mathscr{C}$-morphism (called the **injection from A_j to $\amalg(A_i)_{i\in I}$**).

(3) for each pair $((f_i)_{i\in I}, C)$, (where C is a $\mathscr{C}$-object and for each $j \in I$, f_j: $A_j \to C$) there exists a unique $\mathscr{C}$-morphism (usually denoted by)

$$[f_i]: \amalg(A_i)_{i\in I} \to C$$

such that for each $j \in I$, the triangle



commutes.

18.7 For simplicity one often writes $(\prod_I A_i, (\pi_i))$ or $(\Pi A_i, \pi_i)$ or sometimes even (inaccurately) ΠA_i alone when denoting a product of the family $(A_i)_{i\in I}$. (Dually: $((\mu_i), \coprod_I A_i)$, $(\mu_i, \amalg A_i)$, and $\amalg A_i$.) Also, when for each $i \in I$, $A_i = B$, then ΠA_i is called the **Ith power of B** and is denoted by B^I. In this case the unique morphism $\langle (1_B)_i \rangle: B \to B^I$ is called the **diagonal morphism** and is denoted by

Δ_B, or simply Δ. (Dually: **Ith copower of** B; $^I B$; **codiagonal morphism**; ∇_B; ∇.) Notice that when $I = \{1, 2\}$,

$$(\prod_I A_i, (\pi_i))$$

is "essentially the same" as

$$(A_1 \times A_2, \pi_{A_1}, \pi_{A_2}),$$

so that this justifies the above definition for the generalization of the notion of "product of pairs".

18.8 EXAMPLES

(1) X is a $\mathscr{C}$-terminal object if and only if $(X, \varnothing)$ is a product of an empty indexed family of $\mathscr{C}$-objects.

(2) If A is a $\mathscr{C}$-object, then (P, f) is a product of the self-indexed set $\{A\}$ if and only if $f\colon P \to A$ is a $\mathscr{C}$-isomorphism.

(3) Products and coproducts of arbitrary families of objects in the categories **Set**, **Grp**, R-**Mod**, and **Top** have the same names as those given in 18.4.

(4) The categorical product in the category of abelian torsion groups is not the group-theoretic product but is the torsion subgroup of the group-theoretic product. The coproduct, however, is the direct sum.

(5) The categorical product in the category of locally connected spaces is not the topological product but is the cartesian product of the underlying sets supplied with the coarsest locally connected topology that is finer than the usual product topology. The coproduct in this category, however, is the (disjoint) topological sum.

(6) The coproduct in the category **CompT$_2$** of compact Hausdorff spaces is not the topological sum but is the Stone-Čech compactification of the topological sum. The product in this category, however, is the topological product.

18.9 PROPOSITION (SIMULTANEOUS CANCELLATION)

If $(\Pi A_i, \pi_i)$ *is a product of* $(A_i)_{i \in I}$ *and if* $C \overset{h}{\underset{k}{\rightrightarrows}} \Pi A_i$ *are morphisms with the property that for each* $i \in I$, $\pi_i \circ h = \pi_i \circ k$, then $h = k$; *i.e., "projections, acting in unison, act monomorphically".*

Proof: For each i, let $f_i = \pi_i \circ h = \pi_i \circ k$; then by the uniqueness condition in the definition of product, $h = \langle f_i \rangle = k$. □

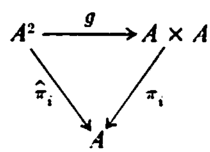
Returning for a moment to the category of sets, consider an infinite set A, the set

$$A \times A = \{(a, b) \mid a, b \in A\},$$

and the set

$$A^2 = \{f\colon \{0, 1\} \to A\}.$$

Now each of $A \times A$ and A^2 is commonly called *the* product of A with itself, yet these sets are quite different. At first glance one might think that they are regarded as "essentially the same" because there is a bijection $g\colon A^2 \to A \times A$ [defined by $g(f) = (f(0), f(1))$]. However, since A is infinite, there is also a bijection from A^2 to A, and clearly A has no claim to the title "the cartesian product of A with itself". What has been forgotten is the fact that each of A^2 and $A \times A$ has projections associated with it: $\hat{\pi}_1\colon A^2 \to A$ defined by $f \mapsto f(0)$; $\hat{\pi}_2\colon A^2 \to A$ defined by $f \mapsto f(1)$; $\pi_1\colon A \times A \to A$ defined by $(a, b) \mapsto a$; and $\pi_2\colon A \times A \to A$ defined by $(a, b) \mapsto b$. The reason that the triples $(A^2, \hat{\pi}_1, \hat{\pi}_2)$ and $(A \times A, \pi_1, \pi_2)$ are thought of as "essentially the same" is because the bijection $g\colon A^2 \to A \times A$ "respects" these projections; i.e., for $i = 1, 2$ the triangle

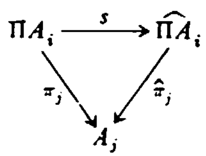


commutes.

A similar "essential uniqueness" holds for any categorical product, as the following proposition shows:

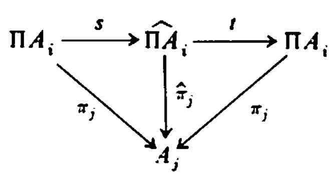
18.10 PROPOSITION (UNIQUENESS OF PRODUCTS)

If each of $(\Pi A_i, \pi_i)$ *and* $(\widehat{\Pi A_i}, \hat{\pi}_i)$ *is a product of* $(A_i)_{i \in I}$ *then there exists a unique isomorphism* $s\colon \Pi A_i \to \widehat{\Pi A_i}$ *such that for each* $j \in I$, *the triangle*



commutes.

Proof: By the definition of product there exist unique morphisms s and t such that for each $j \in I$ the diagram



commutes. Hence for each $j \in I$,

$$\pi_j \circ (t \circ s) = \pi_j = \pi_j \circ 1_{\Pi A_i}.$$

Thus by the cancellation property (18.9), $t \circ s = 1_{\Pi A_i}$. Similarly it can be shown that $s \circ t = 1_{\widehat{\Pi A_i}}$. Consequently, s is an isomorphism. ☐

18.11 COROLLARY

Any two terminal objects of a category are isomorphic (cf. 7.8). ☐

Because of the essential uniqueness of products, one often speaks of *the* product of a family of objects rather than *a* product of the family and similarly for coproducts.

It is well-known that for many concrete categories, the projection morphisms are surjective and the injection morphisms are injective. This is not always the case. For example in **Set** the projection $\mathbf{N} \times \varnothing \to \mathbf{N}$ is the empty function, which is not surjective; and in the category of commutative rings the coproduct $\mathbf{Q} \amalg \mathbf{Z}_2 = \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}_2 = 0$, so that the injection $\mathbf{Q} \to \mathbf{Q} \amalg \mathbf{Z}_2$ is not injective. However, we do have the following:

18.12 PROPOSITION

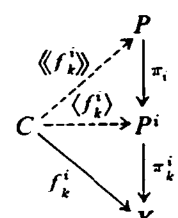
If $\mathscr{C}$ is a connected category, then every projection morphism in $\mathscr{C}$ is a retraction (and dually, every injection morphism is a section).

Proof: Suppose that $\pi_j\colon \Pi A_i \to A_j$ is a projection morphism. For each $i \in I$, $i \neq j$ let $f_i\colon A_j \to A_i$ be any morphism and let $f_j\colon A_j \to A_j$ be 1_{A_j}. Then by the definition of product there exists $\langle f_i \rangle\colon A_j \to \Pi A_i$ such that $\pi_j \circ \langle f_i \rangle = 1_{A_j}$. Hence π_j has a right inverse. □

18.13 PROPOSITION (ITERATION OF PRODUCTS)

Let $(K_i)_{i \in I}$ be a pairwise disjoint family of sets. Suppose that for each $i \in I$ $(P^i, (\pi^i_k)_{k \in K_i})$ is the product of the family $(X_k)_{k \in K_i}$ of $\mathscr{C}$-objects and that (P, π_i) is the product of the family $(P^i)_{i \in I}$. Then $(P, (\pi^i_k \circ \pi_i)_{i \in I, k \in K_i})$ is the product of $(X_k)_{k \in \cup_I K_i}$.

Proof: Clearly for each $i \in I$ and $k \in K_i$, $\pi^i_k \circ \pi_i\colon P \to X_k$. Suppose that C is a $\mathscr{C}$-object and for each $i \in I$ and $k \in K_i$, $f^i_k\colon C \to X_k$. By the definition of product, for each $i \in I$ there exists a unique morphism $\langle f^i_k \rangle\colon C \to P^i$ such that for each $k \in K_i$, $\pi^i_k \circ \langle f^i_k \rangle = f^i_k$. Again by the definition of product, there exists a unique morphism $\langle\langle f^i_k \rangle\rangle\colon C \to P$ such that for each $i \in I$, $\pi_i \circ \langle\langle f^i_k \rangle\rangle = \langle f^i_k \rangle$. Hence for each $i \in I$ and $k \in K_i$ the diagram



commutes. Since each $(P^i, (\pi^i_k))$ is a product, the uniqueness of $\langle\langle f^i_k \rangle\rangle$ is readily established. Hence $(P, \pi^i_k \circ \pi_i)$ is the product of the X_k's. □

18.14 PROPOSITION

If $(\Pi A_i, \pi_i)$ and $(\Pi B_i, \rho_i)$ are products of the families $(A_i)_I$ and $(B_i)_I$, respectively; and if for each $i \in I$ there is a morphism $A_i \xrightarrow{f_i} B_i$, then there exists a unique morphism (usually denoted by) Πf_i that makes each square

$$\begin{array}{ccc} \Pi A_i & \xrightarrow{\Pi f_i} & \Pi B_i \\ {\scriptstyle \pi_j}\downarrow & & \downarrow{\scriptstyle \rho_j} \\ A_j & \xrightarrow[f_j]{} & B_j \end{array}$$

commute. □

18.15 DEFINITION

The morphism Πf_i of the above proposition is called the **product of the morphisms** $(f_i)_I$. If $I = \{1, 2, \ldots, n\}$, Πf_i is sometimes written $f_1 \times f_2 \times \cdots \times f_n$.

DUAL NOTIONS: $\amalg f_i$ is the **coproduct of the morphisms** $(f_i)_I$; $f_1 \amalg f_2 \amalg \cdots \amalg f_n$.

18.16 PROPOSITION

In any category the product of

(1) *retractions is a retraction.*

(2) *sections is a section.*

(3) *isomorphisms is an isomorphism.*

(4) *monomorphisms is a monomorphism.*

(5) *constant morphisms is a constant morphism.*

Proof: For each $i \in I$ let $f_i\colon A_i \to B_i$, let $(\Pi A_i, \pi_i)$ be the product of $(A_i)_I$ and $(\Pi B_i, \rho_i)$ be the product of $(B_i)_I$. Then for each $j \in I$ we have the commutative square

$$\begin{array}{ccc} \Pi A_i & \xrightarrow{\Pi f_i} & \Pi B_i \\ {\scriptstyle \pi_j}\downarrow & & \downarrow{\scriptstyle \rho_j} \\ A_j & \xrightarrow[f_j]{} & B_j \end{array}$$

(*1*). If for each f_j, there is some g_j such that $f_j \circ g_j = 1_{B_j}$, then by pasting together commutative squares we see that for each $j \in I$

$$\rho_j \circ (\Pi f_i) \circ (\Pi g_i) = f_j \circ g_j \circ \rho_j = \rho_j = \rho_j \circ 1_{\Pi B_i}.$$

Hence by the simultaneous cancellation property for the ρ_j's, we have that $(\Pi f_i) \circ (\Pi g_i) = 1_{\Pi B_i}$; i.e., Πf_i is a retraction.

(*2*). Analogous to (1).

(*3*). (1) and (2).

(*4*). If h and k are morphisms such that $(\Pi f_i) \circ h = (\Pi f_i) \circ k$, then for each $j \in I$, $\rho_j \circ (\Pi f_i) \circ h = \rho_j \circ (\Pi f_i) \circ k$, so that for each $j \in I$, $f_j \circ \pi_j \circ h = f_j \circ \pi_j \circ k$. Since each f_j is a monomorphism and since the π_j's in conjunction are left-cancellable (18.9), it follows that $h = k$.

(*5*). Exercise. □

18.17 PROPOSITION

If for each $i \in I$,

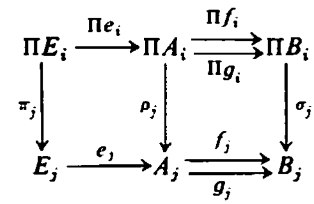
$$(E_i, e_i) \approx Equ(f_i, g_i)$$

and if Πe_i, Πf_i *and* Πg_i *exist, then*

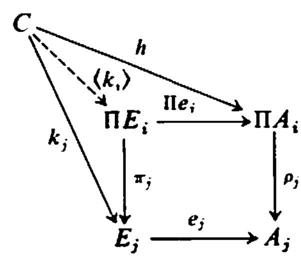
$$(\Pi E_i, \Pi e_i) \approx Equ(\Pi f_i, \Pi g_i).$$

(*I.e., the product of equalizers is an equalizer of the product.*)

Proof: For each $j \in I$, consider the diagram



where π_j, ρ_j, and σ_j are projection morphisms. For each j, $f_j \circ e_j \circ \pi_j = g_j \circ e_j \circ \pi_j$ implies that $\sigma_j \circ \Pi f_i \circ \Pi e_i = \sigma_j \circ \Pi g_i \circ \Pi e_i$, so that by the cancellation property for projections, $\Pi f_i \circ \Pi e_i = \Pi g_i \circ \Pi e_i$. Now suppose that $C \xrightarrow{h} \Pi A_i$ is a morphism such that $\Pi f_i \circ h = \Pi g_i \circ h$. Then for each j, $f_j \circ (\rho_j \circ h) = g_j \circ (\rho_j \circ h)$. Thus since $(E_j, e_j) \approx Equ(f_j, g_j)$, for each j, there exists a unique morphism $C \xrightarrow{k_j} E_j$ such that $e_j \circ k_j = \rho_j \circ h$. By the definition of product, there is a unqiue morphism $\langle k_i \rangle: C \to \Pi E_i$ such that for each j, $k_j = \pi_j \circ \langle k_i \rangle$.



Hence, for each j, $\rho_j \circ \Pi e_i \circ \langle k_i \rangle = \rho_j \circ h$, so that by the cancellation property $\Pi e_i \circ \langle k_i \rangle = h$. Also $\langle k_i \rangle$ is unique with respect to this property because Πe_i is the product of monomorphisms and is thus a monomorphism (18.16). □

18.18 DEFINITION

A category $\mathscr{C}$ **has products** (resp. **has finite products**) provided that for every set (resp. finite set) I, each family of $\mathscr{C}$-objects indexed by I has a $\mathscr{C}$-product.

DUAL NOTIONS: **has coproducts; has finite coproducts.**

18.19 PROPOSITION

In any category $\mathscr{C}$, *the following are equivalent:*

(1) *$\mathscr{C}$ has finite products.*

(2) *$\mathscr{C}$ has a terminal object and a product for each pair of objects in $\mathscr{C}$.* □

18.20 PROPOSITION

In any category that has products, the product of a family of regular monomorphisms is a regular monomorphism.

Proof: See Proposition 18.17. □

18.21 EXAMPLES

(1) The following categories have both products and coproducts: **Set**, **Grp**, ***R*-Mod**, **Top**.

(2) The category **Field** has neither finite products nor finite coproducts. (See Exercise 18L).

(3) **NLinSp** and $\mathbf{BanSp_1}$ each have finite products but not (arbitrary) products. (See Exercise 18M.)

(4) A partially-ordered set (considered as a category) has products if and only if it has coproducts if and only if it is a complete lattice. (Indeed, we have the following:)

18.22 THEOREM (FREYD)

For any small category $\mathscr{C}$, the following are equivalent:

(1) *$\mathscr{C}$ has products.*

(2) *$\mathscr{C}$ has coproducts.*

(3) *$\mathscr{C}$ is equivalent to a complete lattice.*

Proof: If $\mathscr{C}$ is a complete lattice considered as a category, and $(A_i)_I$ is a family of $\mathscr{C}$-objects, it is clear that

$$\prod_I A_i = \inf_I(A_i)$$

and

$$\coprod_I A_i = \sup_I(A_i).$$

Since a complete lattice is a skeletal category, categories are equivalent provided they have isomorphic skeletons, and products are unique only up to commuting isomorphisms, we have that (3) implies (1) and (3) implies (2). If a small category $\mathscr{C}$ has products, then $\mathscr{C}$ is a quasi-ordered set, considered as a category. For suppose to the contrary that $A \underset{h}{\overset{g}{\rightrightarrows}} B$ are distinct $\mathscr{C}$-morphisms. Let I be the cardinal number of $Mor(\mathscr{C})$, and for each function $f: I \to \{0, 1\}$ and each $i \in I$, let

$$r_i^f: A \to B \text{ be } \begin{cases} g & \text{if } f(i) = 0 \\ h & \text{if } f(i) = 1 \end{cases}$$

By the definition of product, for each f, $\langle r_i^f \rangle: A \to B^I$. Now if $f \neq \hat{f}$, then for some $j \in I$, $f(j) \neq \hat{f}(j)$ so that since $g \neq h$, $\pi_j \circ \langle r_i^f \rangle \neq \pi_j \circ \langle r_i^{\hat{f}} \rangle$. Thus

$\langle r_i^f \rangle \neq \langle r_i^{\dot f} \rangle$. Consequently there are at least 2^I different morphisms from A to B^I, which is impossible. Since $\mathscr{C}$ is a quasi-ordered set, any skeleton, $\mathscr{S}$, for it is a partially-ordered set, and, since $\mathscr{C}$ has products, $\mathscr{S}$ has infima. Thus it is a complete lattice. Hence (1) implies (3). Analogously (2) implies (3). ☐

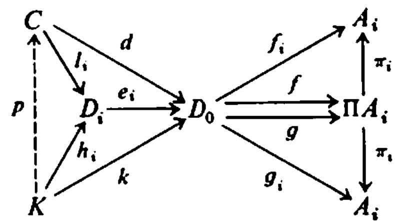
18.23 PROPOSITION

Suppose that for each $i \in I$, $D_i \xrightarrow{e_i} D_0$ is the equalizer of $D_0 \underset{g_i}{\overset{f_i}{\rightrightarrows}} A_i$, $(\Pi A_i, (\pi_i))$ is the product of the family $(A_i)_I$ and $f = \langle f_i \rangle \colon D_0 \to \Pi A_i$ and $g = \langle g_i \rangle \colon D_0 \to \Pi A_i$ are the unique morphisms induced by the product. Then (C, d) is the equalizer of f and g if and only if it is the intersection of the family $(D_i, e_i)_I$.

Proof: If $(C, d) \approx Equ(f, g)$, then for each i, $f_i \circ d = g_i \circ d$ implies that there exists a unique morphism $l_i \colon C \to D_i$ such that $d = e_i \circ l_i$. Now suppose that $(K \xrightarrow{h_i} D_i)$ and $K \xrightarrow{k} D_0$ are morphisms such that for each i, $e_i \circ h_i = k$. Then for each i,

$$\pi_i \circ g \circ k = g_i \circ e_i \circ h_i = f_i \circ e_i \circ h_i = \pi_i \circ f \circ k,$$

so that by the cancellation property for projections, $g \circ k = f \circ k$. Hence since (C, d) is the equalizer of f and g, there exists a unique morphism $p \colon K \to C$ such that $d \circ p = k$. Thus (C, d) is the intersection of the family $(D_i, e_i)_I$.



Conversely, if (C, d) is the intersection of the family $(D_i, e_i)_I$, then for each i there is a morphism $l_i \colon C \to D_i$ such that $d = e_i \circ l_i$. Hence, for each i,

$$\pi_i \circ f \circ d = f_i \circ e_i \circ l_i = g_i \circ e_i \circ l_i = \pi_i \circ g \circ d.$$

Again by the cancellation property for projections, we have $f \circ d = g \circ d$. Now suppose that k is a morphism such that $f \circ k = g \circ k$. This implies that for each i, $f_i \circ k = g_i \circ k$ so that there exist morphisms h_i such that $e_i \circ h_i = k$. But by the definition of intersection, there exists a unique morphism $\bar{p}$ such that

$$d \circ \bar{p} = k.$$

Thus $(C, d) \approx Equ(f, g)$. ☐

18.24 COROLLARY

If $\mathscr{C}$ has products, then every intersection of regular subobjects in $\mathscr{C}$ is a regular subobject. ☐

18.25 COROLLARY

If $\mathscr{C}$ has products (resp. finite products) and equalizers, then $\mathscr{C}$ has intersections (resp. finite intersections) of regular subobjects. □

EXERCISES

18A. If the category $\mathscr{C}$ has finite products, show that there is a bifunctor $\_ \times \_ : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ defined by:

$$\_ \times \_(A, B) = A \times B,$$

and if

$$f: A \to A' \quad \text{and} \quad g: B \to B'$$

$$\_ \times \_(f, g) = f \times g: A \times B \longrightarrow A' \times B',$$

where $f \times g$ is the unique morphism which makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & A' \\ {\scriptstyle \pi_A}\uparrow & & \uparrow{\scriptstyle \pi_{A'}} \\ A \times B & \xrightarrow{\ f \times g\ } & A' \times B' \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi_{B'}} \\ B & \xrightarrow[\quad g \quad]{} & B' \end{array}$$

commute. Dually, if $\mathscr{C}$ has finite coproducts, show that there is a bifunctor $\_ \amalg \_ :$ $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$, which assigns to each pair of $\mathscr{C}$-objects (A, B) the coproduct $A \amalg B$. (Cf. 9.8 and 9I.)

18B. Suppose that $\mathscr{C}$ has finite products, $B \in Ob(\mathscr{C})$, and $\_ \times B: \mathscr{C} \to \mathscr{C}$ is the left associated functor (with respect to B) of the functor described in 18A. Show that the family $(\pi_A)_{A \in Ob(\mathscr{C})}$ is a natural transformation from $\_ \times B$ to $1_{\mathscr{C}}$.

18C. Show that if $\mathscr{B}$ has products and if $F: \mathscr{C} \to \mathscr{B}$, then for each set I, there is an induced functor $F^I: \mathscr{C} \to \mathscr{B}$, defined by

$$F^I(A) = (F(A))^I,$$

for each $A \in Ob(\mathscr{C})$.

$$F^I(f) = \prod_I F(f): (F(A))^I \to (F(B))^I,$$

for each $A \xrightarrow{f} B \in Mor(\mathscr{C})$.

18D. If $(A_i)_{i \in I}$ is a set-indexed family of $\mathscr{C}$-objects, prove that the following are equivalent:

(a) $(P, (\pi_i))$ is a $\mathscr{C}$-product of (A_i).

(b) For each $B \in Ob(\mathscr{C})$, the function $hom_{\mathscr{C}}(B, P) \to \prod_I hom_{\mathscr{C}}(B, A_i)$, defined by:

$$B \xrightarrow{f} P \mapsto (\pi_i \circ f)_{i \in I}$$

is bijective (where $\prod_I hom(B, A_i)$ is the cartesian product in Set and $(\pi_i \circ f)_{i \in I}$ is the unique element of the cartesian product whose ith coordinate is $\pi_i \circ f$). Conclude that categorical products can be defined in terms of cartesian products of sets (cf. 6B).

18E. Suppose that $(P, (\pi_i))$ is the product in $\mathscr{C}$ of the family (A_i) and that $f: B \to P$.
(a) Prove that f is a constant morphism if and only if $\pi_i \circ f$ is a constant morphism for each i.
(b) Show that if there is some i such that $\pi_i \circ f$ is a monomorphism (resp. section), then f is a monomorphism (section).

18F. Suppose that A is a $\mathscr{C}$-object, K and L are sets such that $\varnothing \neq K \subset L$ and A^K and A^L are powers of A. Show that A^K can be considered as being simultaneously a subobject of A^L and a quotient object of A^L by exhibiting a section s and a retraction r such that

$$A^K \xrightarrow{s} A^L \xrightarrow{r} A^K = 1_{A^K}.$$

18G. Show that the product of epimorphisms is not necessarily an epimorphism.

18H. Prove that the product of multiple equalizers is the multiple equalizer of the product (cf. 18.17).

18I. Let X be a $\mathscr{C}$-terminal object. For each $\mathscr{C}$-object A prove that the product (object) $X \times A$ exists and is isomorphic to A.

18J. Show that X is a terminal object in **Set** if and only if each **Set**-object is a copower of X.

18K. Show that any non-trivial group considered as a category does not have finite products.

18L. Let $\mathscr{C}$ be a category such that:

(α) every morphism is a monomorphism; and
(β) there exist two distinct morphisms with the same domain and the same codomain.

(a) Prove that $\mathscr{C}$ does not have finite products.
(b) Prove that $\mathscr{C}$ does not have finite coproducts.
(c) Conclude that **Field** has neither finite products nor finite coproducts.

18M. Let A and B be Banach spaces.

(a) If P is the cartesian product of A and B supplied with the sup-norm (i.e., $\|(a, b)\| = sup\{\|a\|, \|b\|\}$), show that (P, π_A, π_B) is a product of A and B in **NLinSp**, **BanSp$_1$**, and **BanSp$_2$**.
(b) If Q is the cartesian product of A and B supplied with the sum-norm (i.e., $\|(a, b)\| = \|a\| + \|b\|$), show that (Q, π_A, π_B) is a product of A and B in **NLinSp** and in **BanSp$_1$** but is not a product of A and B in **BanSp$_2$**.
(c) Show that neither of the categories **NLinSp** and **BanSp$_1$** has products.

§19 SOURCES AND SINKS

As we have seen, the product of a family $(X_i)_I$ of objects in a category is defined to be a pair $(\Pi X_i, (\pi_i)_I)$; where ΠX_i is an object and $(\pi_i)_I$ is a family of morphisms with common domain ΠX_i, satisfying certain conditions (such as acting "in concert" monomorphically). Such families of morphisms with common domain (or dually with common codomain) appear frequently and thus deserve special attention. In this section we will provide some of the necessary details that will look rather technical at first sight, but that will turn out to be very useful.

Mono-Sources and Epi-Sinks

19.1 DEFINITION

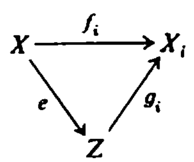
(1) A **source in $\mathscr{C}$** is a pair $(X, (f_i)_I)$, where X is a $\mathscr{C}$-object and $(f_i: X \to X_i)_I$ is a family of $\mathscr{C}$-morphisms each with domain X. In this case X is called the **domain of the source** and the family $(X_i)_I$ is called the **codomain of the source.** [To simplify notation a source $(X, (f_i)_I)$ is often denoted by (X, f_i).]

(2) A source (X, f_i) is called a **mono-source** provided that the f_i can be simultaneously cancelled from the left; i.e., provided that for any pair $Y \overset{r}{\underset{s}{\rightrightarrows}} X$ of morphisms such that $f_i \circ r = f_i \circ s$ for each i, it follows that $r = s$.

(3) A source (X, f_i) is called an **(extremal mono)-source** provided that

(i) it is a mono-source, and

(ii) (*Extremal condition*): for each source (Z, g_i) and each epimorphism e such that for each i, the triangle



commutes, e must be an isomorphism.

DUAL NOTIONS: **sink in $\mathscr{C}$**; (f_i, X); **codomain of a sink**; **domain of a sink**; **epi-sink**; **(extremal epi)-sink.**

19.2 EXAMPLES

(1) $(X, \varnothing)$ is a mono-source if and only if for each object Y there is at most one morphism from Y to X. (Hence, in case the category is connected, this is equivalent to the condition that X is a terminal object.)

(2) (X, f) is a mono-source if and only if f is a monomorphism. It is an (extremal mono)-source if and only if f is an extremal monomorphism.

(3) Each product $(\Pi X_i, \pi_i)$ is a mono-source (18.9). In fact each product is an (extremal mono)-source (Exercise 19D).

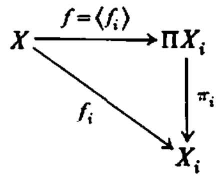
(4) Let (f_i, X) be a sink in one of the categories **Grp**, **SGrp**, or R-**Mod**. Then (f_i, X) is an (extremal epi)-sink if and only if the union of the set-theoretic images of the homomorphisms f_i generates X in the usual algebraic sense.

(5) Let (X, f_i) be a source in **Top**. If (X, f_i) is an (extremal mono)-source, then X has the weak (i.e., coarse) topology with respect to the continuous functions f_i. Conversely if (X, f_i) is a mono-source, then it is an (extremal mono)-source if X has the weak topology with respect to the functions f_i.

(6) Let (f_i, X) be a sink in **Top**. If (f_i, X) is an (extremal epi)-sink, then X has the strong (i.e., fine) topology with respect to the continuous functions f_i. Conversely if (f_i, X) is an epi-sink, then X has the strong topology with respect to the functions f_i only if (f_i, X) is an (extremal epi)-sink.

19.3 PROPOSITION

Let (X, f_i) be a source, $(\Pi X_i, \pi_i)$ be the product of the codomain (X_i) of the source, and $f: X \to \Pi X_i$ be the unique induced morphism for which all triangles

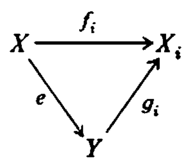


commute. Then

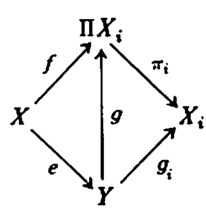
(1) *(X, f_i) is a mono-source if and only if f is a monomorphism.*

(2) *(X, f_i) is an (extremal mono)-source if and only if f is an extremal monomorphism.*

Proof: (1) is immediate since $(\Pi X_i, \pi_i)$ is a mono-source. It is also clear that if (X, f_i) is an (extremal mono)-source, then f is an extremal monomorphism. To show the converse, suppose that f is an extremal monomorphism, (Y, g_i) is a source and e is an epimorphism such that for each i the triangle



commutes. By the definition of product, there is a unique morphism $g: Y \to \Pi X_i$ such that for each i, $g_i = \pi_i \circ g$. Now since $(\Pi X_i, \pi_i)$ is a mono-source, this implies that the diagram



commutes. Thus since f is an extremal monomorphism, e must be an isomorphism. □

19.4 PROPOSITION

If $\mathscr{C}$ has coequalizers and (X, f_i) is a source in $\mathscr{C}$ which satisfies the extremal condition (ii) *of Definition* 19.1(3), *then (X, f_i) must be an (extremal mono)-source.*

Proof: Analogous to the proof of Proposition 17.14 dual. □

Separators and Coseparators

19.5 PROPOSITION

For any $\mathscr{C}$-object C, the following are equivalent:

(1) *C is a coseparator.*

(2) *For each $\mathscr{C}$-object X, the source $(X, hom(X, C))$ is a mono-source.* □

19.6 PROPOSITION

If $\mathscr{C}$ has arbitrary powers of the object C, then the following are equivalent:

(1) *C is a coseparator.*

(2) *For each $\mathscr{C}$-object X, the unique morphism induced by the product,*

$$X \to C^{hom(X,C)}$$

is a monomorphism.

(3) *Each $\mathscr{C}$-object is a subobject of some power C^I of C.*

Proof: If C is a coseparator, then by the above proposition (19.5) $(X, hom(X, C))$ is a mono-source, so that the induced morphism $X \to C^{hom(X,C)}$ must be a monomorphism (19.3). Hence (1) implies (2). Clearly (2) implies (3). To show that (3) implies (1), let X be any $\mathscr{C}$-object. Then by hypothesis, there is an index set I and a monomorphism $X \xrightarrow{f} C^I$. But $f = \langle \pi_i \circ f \rangle$, so that by Proposition 19.3, $(X, \pi_i \circ f)$ is a mono-source. But each $\pi_i \circ f$ belongs to $hom(X, C)$. Hence $(X, hom(X, C))$ is a mono-source, so that by the above proposition, (19.5), C is a coseparator for $\mathscr{C}$. □

19.7 DEFINITION

Let $\mathscr{C}$ be a category which has products and let $\mathscr{M}$ be a class of monomorphisms in $\mathscr{C}$.

A $\mathscr{C}$-object C is called an **$\mathscr{M}$-coseparator of $\mathscr{C}$** provided that each $\mathscr{C}$-object is an $\mathscr{M}$-subobject of a suitable power C^I of C. In particular: (extremal monomorphism)-coseparators are called **extremal coseparators** and (regular monomorphism)-coseparators are called **regular coseparators**. (Cf. 19.6.)

DUAL NOTIONS: (if $\mathscr{C}$ has coproducts and $\mathscr{E}$ is a class of epimorphisms in $\mathscr{C}$): **$\mathscr{E}$-separator**; **extremal separator**; **regular separator.**

19.8 EXAMPLES

(1) The closed unit interval is both an extremal separator and an extremal coseparator for $\mathbf{CompT_2}$.

(2) The two-element discrete space is both an extremal separator and an extremal coseparator for the category of zero-dimensional compact Hausdorff spaces.

(3) $\mathbf{CRegT_2}$ has no extremal coseparator.

(4) For the categories **Grp**, R-**Mod**, and **Set**, the coseparators are precisely the same as the extremal coseparators.

Stronger Smallness Conditions

19.9 DEFINITION

A category $\mathscr{C}$ is called **strongly well-powered** provided that for each set-indexed family $(X_i)_I$ of $\mathscr{C}$-objects, there is at most a set of pairwise non-isomorphic $\mathscr{C}$-objects X with the property that there is a mono-source from X to $(X_i)_I$.

DUAL NOTION: **strongly co-(well-powered).**

19.10 EXAMPLES

Set, **Top**, and **Grp** are both strongly well-powered and strongly co-(well-powered).

19.11 PROPOSITION

Every strongly well-powered category is well-powered.

Proof: Consider the one-element families. □

For a category the properties of being well-powered and strongly well-powered are, in general, different (see Exercise 19F dual). However, as the next proposition shows, under the often satisfied condition that the category has products, the two properties are equivalent.

19.12 PROPOSITION

If a category $\mathscr{C}$ has products, then $\mathscr{C}$ is well-powered if and only if it is strongly well-powered.

Proof: Suppose that $\mathscr{C}$ is well-powered and $(X_i)_I$ is a set-indexed family of $\mathscr{C}$-objects. Let $(\Pi X_i, \pi_i)$ be the product of the family $(X_i)_I$. Then by the definition of product, for each mono-source $(A, A \xrightarrow{f_i} X_i)$ with codomain $(X_i)_I$, there exists a unique morphism $f: A \to \Pi X_i$ such that for each $i \in I$, $f_i = \pi_i \circ f$. Also, by Proposition 19.3, f is a monomorphism. Thus for each mono-source $(A, (f_i))$ with codomain $(X_i)_I$, there corresponds a subobject (A, f) of ΠX_i. Since $\mathscr{C}$ is well-powered, there is no more than a set of pairwise non-isomorphic subobjects of ΠX_i. Hence there is no more than a set of pairwise non-isomorphic $\mathscr{C}$-objects A with the property that there is a mono-source from A to $(X_i)_I$. □

Factorizations of Sources and Sinks

19.13 DEFINITION

Let $(X_i \xrightarrow{f_i} X, X)$ be a sink in $\mathscr{C}$. Then $X_i \xrightarrow{g_i} Y \xrightarrow{m} X$ is called an **(epi-sink, mono)-factorization of** (f_i, X) if and only if:

(1) (g_i, Y) is an epi-sink,

(2) $m: Y \to X$ is a monomorphism, and

(3) for each i, $X_i \xrightarrow{f_i} X = X_i \xrightarrow{g_i} Y \xrightarrow{m} X$.

Analogously, one has **[(extremal epi)-sink, mono]-factorizations** and **(epi-sink, extremal mono)-factorizations.**

DUAL NOTIONS: **(epi, mono-source)-factorization; [epi, (extremal mono)-source]-factorizations; (extremal epi, mono-source)-factorizations.**

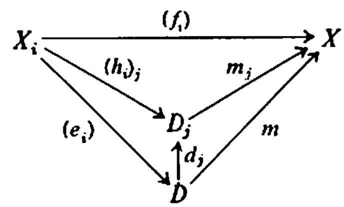
19.14 PROPOSITION

If $\mathscr{C}$ is well-powered and has intersections and equalizers, then every sink in $\mathscr{C}$ has an [(extremal epi)-sink, mono]-factorization.

Proof: The proof is analogous to the proofs of Propositions 17.8 and 17.16. Let (f_i, X) be a sink in $\mathscr{C}$ and let $(D_j, m_j)_J$ be the family of all subobjects of X that are part of some factorization

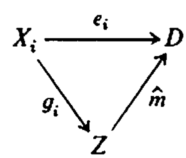
$$X_i \xrightarrow{(f_i)} X = X_i \xrightarrow{(h_i)_j} D_j \xrightarrow{m_j} X$$

of the sink (f_i, X). By Proposition 17.7, the family $(D_j, m_j)_J$ has an intersection (D, m) that is a subobject of X. By the definition of intersection, for each i, there is a unique morphism e_i: $X_i \to D$ such that for each j the diagram



commutes.

We need only show that (e_i, D) is an (extremal epi)-sink, and to do this it is sufficient to show that it satisfies the dual of the extremal condition (ii) of Definition 19.1(3) (see Proposition 19.4 dual). Suppose that (g_i, Z) is a sink and $\hat{m}$ is a monomorphism such that for each i, the triangle



commutes. Then $(Z, m \circ \hat{m})$ belongs to the family $(D_j, m_j)_J$; i.e., there is some $\hat{j} \in J$ such that $m \circ \hat{m} \circ d_{\hat{j}} = m = m \circ 1$. Since m is a monomorphism, we have $\hat{m} \circ d_{\hat{j}} = 1$. Thus $\hat{m}$ is a retraction (and a monomorphism); hence an isomorphism. Consequently (e_i, D) is an (extremal epi)-sink. □

EXERCISES

19A. Suppose that (f_i, X) is a sink and for each i, $f_i = f$. Prove that (f_i, X) is an epi-sink if and only if f is an epimorphism.

19B. Suppose that one has the factorization

$$A \xrightarrow{(f_i)} B_i = A \xrightarrow{h} C \xrightarrow{(g_i)} B_i$$

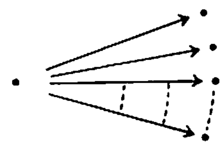
of the source (A, f_i). Prove the following:

(a) If (A, f_i) is a mono-source (resp. (extremal mono)-source), then h is a monomorphism (resp. extremal monomorphism).

(b) If h is a monomorphism and (C, g_i) is a mono-source, then (A, f_i) is a mono-source. Obtain some earlier results about monomorphisms as corollaries.

19C. Interpret a source in $\mathscr{C}$ in the following two ways:

(a) as a functor from a category of the form



into $\mathscr{C}$.
(b) as a natural transformation between two functors from a discrete category into $\mathscr{C}$.

19D. Prove that each product $(\Pi X_i, \pi_i)$ is an (extremal mono)-source.

19E. Prove that if $\mathscr{C}$ has products and C is a $\mathscr{C}$-object, then the following are equivalent:

(a) C is an extremal coseparator.
(b) For each object X, $(X, hom(X, C))$ is an (extremal mono)-source.
(c) For each $\mathscr{C}$-object X, the unique morphism induced by the product $X \to C^{hom(X,C)}$ is an extremal monomorphism.

19F. Consider the following subcategory $\mathscr{C}$ of **Set**. Let a be a set and for each ordinal α let α_1 and α_2 be unequal sets such that for any two ordinals α and $\hat{\alpha}$, $\alpha_i \neq \hat{\alpha}_j$; $i, j = 1, 2$.

The objects of $\mathscr{C}$ are the following sets: $\{a\}$ and A_α, for each ordinal α, where $A_\alpha = \{\alpha_1, \alpha_2\}$. Besides the identity functions, there are the following *hom*-sets for each α.

$$hom(A_\alpha, A_\alpha) = \text{all four functions from } A_\alpha \text{ to } A_\alpha.$$
$$hom(\{a\}, A_\alpha) = \text{both functions from } \{a\} \text{ to } A_\alpha.$$

All *hom*-sets not specified by the above are empty.
(a) Prove that in $\mathscr{C}$ the epimorphisms are precisely the surjections, and conclude that $\mathscr{C}$ is a co-(well-powered) category.
(b) Prove that $\mathscr{C}$ is not strongly co-(well-powered).

19G. Determine whether or not the category of complete lattices (and complete homomorphisms) is either co-(well-powered) or strongly co-(well-powered).

19H. In a category that has equalizers, show that a sink (f_i, A) is an epi-sink if and only if whenever there is a factorization of (f_i, A)

$$X_i \xrightarrow{f_i} A = X_i \xrightarrow{g_i} B \xrightarrow{m} A$$

with m a regular monomorphism, m must be an isomorphism. Obtain 16G as a corollary.

19I. Obtain Proposition 17.16 as a corollary to Proposition 19.14.

19J. *Separating Sets*
Let $\mathscr{C}$ be any category.

(a) Prove that for each set $\mathscr{S}$ of $\mathscr{C}$-objects, the following conditions are equivalent:

(i) For any pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ of distinct $\mathscr{C}$-morphisms, there exists some $S \in \mathscr{S}$ and $h\colon S \to X$ such that $f \circ h \neq g \circ h$.

(ii) For each $\mathscr{C}$-object X, the pair $(\bigcup_{S \in \mathscr{S}} hom(S, X), X)$ is an epi-sink.

$\mathscr{S}$ is called a **separating set for** $\mathscr{C}$ provided that it satisfies (i) and (ii).

(b) Prove that if $\mathscr{C}$ is a connected category that has coproducts and $\mathscr{S}$ is a set of $\mathscr{C}$-objects, then $\mathscr{S}$ is a separating set for $\mathscr{C}$ if and only if $\amalg\{S \mid S \in \mathscr{S}\}$ is a separator for $\mathscr{C}$.

(c) Show that **Set** $\times$ **Set** has no separator, but that

$$\{(\varnothing, \{\varnothing\}), (\{\varnothing\}, \varnothing)\}$$

is a separating set for **Set** $\times$ **Set**.

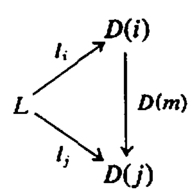
19K. Prove that the functor $\_ \times \_$: **Set** $\times$ **Set** $\to$ **Set** (18A) is not faithful.

§20 LIMITS AND COLIMITS

In this section we introduce the notion of the limit of a functor, which is a generalization of each of the notions "terminal object", "equalizer", "intersection", and "product". We have seen that using the categorical notion of product, one can simultaneously prove many theorems about particular products in various categories. Generalizing one step further, we will see that with the concept of limit one can simultaneously prove theorems about particular limits (such as equalizers, intersections, and products). For example, the fact that limits are "essentially unique" (20.6) will tell us immediately that terminal objects, equalizers, intersections, and products are "essentially unique". Later, other special types of limits, such as inverse limits of directed systems, inverse images, and pullbacks will be introduced and will provide us with additional useful tools for working within the realm of category theory.

20.1 DEFINITION

If I and $\mathscr{C}$ are categories and $D\colon I \to \mathscr{C}$ is a functor, then a **natural source for** D is a source $(L, (l_i)_{i \in Ob(I)})$ in $\mathscr{C}$ such that for each $i \in Ob(I)$, $l_i\colon L \to D(i)$; and for all morphisms $m\colon i \to j$ in I, the triangle



commutes.

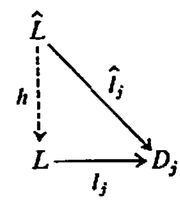
In other words, if $L\colon I \to \mathscr{C}$ is the constant functor whose value at each object is L and whose value at each morphism is 1_L, and if $(L, (l_i)_{i \in Ob(I)})$ is a source in $\mathscr{C}$, then $(L, (l_i)_{i \in Ob(I)})$ is a natural source for D if and only if $(l_i)_{i \in Ob(I)}$ is a natural transformation from L to D.

DUALLY: A **natural sink for** D is a sink $((k_i)_{i \in Ob(I)}, K)$ where $(k_i)_{i \in Ob(I)}$ is a natural transformation from D to the constant functor $K: I \to \mathscr{C}$.

To simplify notation, we often write D_i rather than $D(i)$ and usually write $(L, (l_i)_I)$ or even (L, l_i) to denote the natural source $(L, (l_i)_{i \in Ob(I)})$.

20.2 DEFINITION

If $D: I \to \mathscr{C}$ is a functor, then a natural source (L, l_i) for D is called a **limit of** D provided that if $(\hat{L}, \hat{l}_i)$ is any natural source for D, then there is a *unique* morphism $h: \hat{L} \to L$ such that for each $j \in Ob(I)$, the triangle



commutes [i.e., provided that (L, l_i) is a "terminal" natural source (see Exercise 20A)].

DUALLY: A natural sink (k_i, K) is called a **colimit of** D provided that every natural sink for D factors uniquely through it.

20.3 EXAMPLES

(1) Let I be the category

$$\overset{1}{\bullet} \underset{n}{\overset{m}{\rightrightarrows}} \overset{2}{\bullet},$$

and let $D: I \to \mathscr{C}$. Then $(L, (l_i)_{i=1,2})$ is a limit of D if and only if (L, l_1) is an equalizer of $D(m)$ and $D(n)$ and $l_2 = D(m) \circ l_1 = D(n) \circ l_1$. $((k_i)_{i=1,2}, K)$ is a colimit of D if and only if (k_2, K) is a coequalizer of $D(m)$ and $D(n)$ and $k_1 = k_2 \circ D(m) = k_2 \circ D(n)$.

(2) Let I be a category that is just a sink $(A_i \xrightarrow{f_i} A_0, A_0)$, and let $D: I \to \mathscr{C}$ be a functor such that for each i, $D(f_i)$ is a monomorphism. Then $(L, (l_i), l_0)$ is a limit of D if and only if (L, l_0) is an intersection of the family $(D(A_i), D(f_i))$ of subobjects of $D(A_0)$ and $l_0 = D(f_i) \circ l_i$ for each $i \in I$.

(3) Let I be any discrete category, and let $D: I \to \mathscr{C}$, then $(L, (l_i))$ is a limit of D if and only if it is a product of the family $(D(i))_{i \in Ob(I)}$, and $((k_i), K)$ is a colimit of D if and only if it is a coproduct of the family $(D(i))_{i \in Ob(I)}$.

(4) If I is the empty category and $D: I \to \mathscr{C}$, then $(L, (l_i))$ is a limit of D if and only if L is a terminal object for $\mathscr{C}$ and $(l_i) = \varnothing$; likewise $((k_i), K)$ is a colimit of D if and only if K is an initial object for $\mathscr{C}$ and $(k_i) = \varnothing$.

(5) If $D: \mathscr{C} \to \mathscr{C}$ is the identity functor, then $(L, (l_A))$ is a limit of D if and only if L is an initial object of $\mathscr{C}$ and for each $A \in Ob(\mathscr{C})$, l_A is the unique morphism

from L to A. $((k_A), K)$ is a colimit of D if and only if K is a terminal object of $\mathscr{C}$ and for each $A \in Ob(\mathscr{C})$, k_A is the unique morphism from A to K. [Notice that in this example the families $(l_A)_{A \in Ob(\mathscr{C})}$ and $(k_A)_{A \in Ob(\mathscr{C})}$ are not necessarily sets.]

20.4 PROPOSITION

Any limit $(L, (l_i))$ of a functor $D: I \to \mathscr{C}$, is an (extremal mono)-source.

Proof: If $Q \overset{r}{\underset{s}{\rightrightarrows}} L$ are $\mathscr{C}$-morphisms such that for each $i \in Ob(I)$,

$$l_i \circ r = l_i \circ s,$$

then $(Q, (l_i \circ r)_{i \in Ob(I)})$ is a natural source for D. Thus by the definition of limit, there is a unique morphism $Q \xrightarrow{h} L$ such that for each $i \in Ob(I)$,

$$l_i \circ h = l_i \circ r.$$

But each of r and s is such an h. Hence $r = s$. Consequently $(L, (l_i))$ is a mono-source. To show that it is extremal, suppose that it has factorization

$$L \xrightarrow{l_i} D(i) = L \xrightarrow{e} R \xrightarrow{f_i} D(i),$$

where e is an epimorphism. Since e is an epimorphism, $(R, (f_i))$ is clearly a natural source for D. Hence by the definition of limit, there is a morphism $g: R \to L$ such that for each $i, f_i = l_i \circ g$. Hence, for each i

$$l_i \circ g \circ e = f_i \circ e = l_i = l_i \circ 1$$

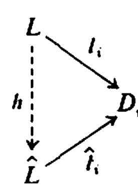
so that since $(L, (l_i))$ is a mono source, $g \circ e = 1$. Thus e is a section (and an epimorphism); hence e is an isomorphism. □

20.5 COROLLARY

Each regular monomorphism is an extremal monomorphism, and each product is an (extremal mono)-source (cf. 17.11 dual and Exercise 19D). □

20.6 PROPOSITION (UNIQUENESS OF LIMITS)

If each of $(L, (l_i))$ and $(\hat{L}, (\hat{l}_i))$ is a limit of the functor $D: I \to \mathscr{C}$, then there exists a unique isomorphism $h: L \to \hat{L}$ such that for each $i \in Ob(I)$ the triangle



commutes.

Proof: By the definition of limit there are unique morphisms $h: L \to \hat{L}$ and $k: \hat{L} \to L$ such that for all i, $\hat{l}_i \circ h = l_i$ and $l_i \circ k = \hat{l}_i$. Consequently, for all i,

$$l_i \circ k \circ h = \hat{l}_i \circ h = l_i = l_i \circ 1_L$$

so that since $(L, (l_i))$ is a mono-source, $k \circ h = 1_L$. Similarly $h \circ k = 1_{\hat{L}}$; hence h is an isomorphism. □

20.7 COROLLARY

Terminal objects, equalizers, multiple equalizers, intersections, and products are "essentially unique" (cf. 7.8, 16.5, 16.12, 17.4, 18.10). □

20.8 Because of the essential uniqueness of limits, by an abuse of the language we often speak of *the* limit of a functor $D: I \to \mathscr{C}$ (when one exists) and denote it by *Lim D*. Dually we speak of *the* colimit of D (denoted by *Colim D*). Thus in general we write

$$Lim\ D \approx (L, (l_i))$$

and

$$Colim\ D \approx ((k_i), K).$$

However, we sometimes (inaccurately) call the object L the limit of D and write $Lim\ D \approx L$. For example, when I is discrete,

$$Lim\ D \approx (\Pi D(i), \pi_i)$$

and

$$Colim\ D \approx (\mu_i, \amalg D(i));$$

when I is $\bullet \underset{n}{\overset{m}{\rightrightarrows}} \bullet$,

$$Lim\ D \approx Equ(D(m), D(n))$$

and

$$Colim\ D \approx Coeq(D(m), D(n))$$

and when I is the empty category, *Lim D* is the terminal object and *Colim D* is the initial object.

EXERCISES

20A. *The Category of Natural Sources*
Suppose that $D: I \to \mathscr{C}$ is a functor. Let $\mathscr{C}_D$ be the quasicategory whose objects are the natural sources for D where the morphisms from (L, l_i) to $(\hat{L}, \hat{l}_i)$ are precisely those morphisms $f: L \to \hat{L}$ such that for each i, $l_i = \hat{l}_i \circ f$; and composition of morphisms is that induced from $\mathscr{C}$.
(a) Show that $\mathscr{C}_D$ is indeed a quasicategory.
(b) Prove that (L, l_i) is a terminal object of $\mathscr{C}_D$ if and only if it is a limit of D.
(c) Obtain as immediate corollaries the facts that limits are mono-sources and are essentially unique.
(By the above, every limit is a terminal object. We have already seen that each terminal object is a limit. Thus in some sense the theory of limits is equivalent to the theory of terminal objects.)

20B. Describe multiple equalizers as limits and multiple coequalizers as colimits.

20C. Suppose that $D: I \to \mathscr{C}$ is a functor.
(a) Prove that the following are equivalent:
(i) $(L, (l_i))$ is a natural source for D.
(ii) $((l_i), L)$ is a natural sink for $D^{op}: I^{op} \to \mathscr{C}^{op}$.

(b) and that the following are equivalent:

(i) $(L, (l_i))$ is a limit of D.

(ii) $((l_i), L)$ is a colimit of D^{op}.

20D. *General Comma Categories*

If $\mathscr{A} \xrightarrow{F} \mathscr{C}$ and $\mathscr{B} \xrightarrow{G} \mathscr{C}$ are functors, then the **comma category** (F, G) is the category whose objects are those triples of the form (A, f, B) where $A \in Ob(\mathscr{A})$, $B \in Ob(\mathscr{B})$, and $f\colon F(A) \to G(B)$; and whose morphisms are those pairs

$$(a, b)\colon (A, f, B) \to (A', f', B')$$

where $a\colon A \to A'$ and $b\colon B \to B'$; such that the square

$$\begin{array}{ccc} F(A) & \xrightarrow{f} & G(B) \\ {\scriptstyle F(a)}\downarrow & & \downarrow{\scriptstyle G(b)} \\ F(A') & \xrightarrow{f'} & G(B') \end{array}$$

commutes. Composition of morphisms is defined by:

$$(a, b) \circ (\hat{a}, \hat{b}) = (a \circ \hat{a}, b \circ \hat{b}).$$

(a) Verify that (F, G) is indeed a category.

(b) If $F\colon \mathscr{C} \to \mathscr{C}$ is the identity functor and $G\colon \mathbf{1} \to \mathscr{C}$ is the functor whose value at the single object is A, show that (F, G) is isomorphic with the comma category $(\mathscr{C}, A)$ and that (G, F) is isomorphic with the comma category $(A, \mathscr{C})$ (4.18 and 4.19).

(c) If each of F and G is the identity functor on $\mathscr{C}$, show that (F, G) is isomorphic with the arrow category $\mathscr{C}^2$ (4.16).

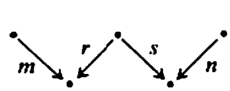
(d) If $F\colon \mathbf{1} \to \mathscr{C}$ is the functor whose value at the single object is A and $G\colon \mathbf{1} \to \mathscr{C}$ is the functor whose value at the single object is B, show that (F, G) is isomorphic with the discrete category (i.e., the set) $hom_{\mathscr{C}}(A, B)$.

(e) Let $P_1, P_2\colon \mathscr{C}^2 \to \mathscr{C}$ be functors defined by $P_1(f, g) = f$ and $P_2(f, g) = g$. Define "projection functors" $Q_1\colon (F, G) \to \mathscr{A}$, $Q_2\colon (F, G) \to \mathscr{B}$, and $H\colon (F, G) \to \mathscr{C}^2$ such that the diagram

$$\begin{array}{ccccc} \mathscr{A} & \xleftarrow{Q_1} & (F, G) & \xrightarrow{Q_2} & \mathscr{B} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle G} \\ \mathscr{C} & \xleftarrow[P_1]{} & \mathscr{C}^2 & \xrightarrow[P_2]{} & \mathscr{C} \end{array}$$

commutes.

(f) Let I be the category



and let $D\colon I \to \mathscr{C}\mathscr{A}\mathscr{T}$ be the functor defined by $D(m) = F$, $D(n) = G$, $D(r) = P_1$, and $D(s) = P_2$. Show that (F, G) together with the functors Q_1, $P_1 \circ H$, H, $P_2 \circ H$, and Q_2 is the limit of D.

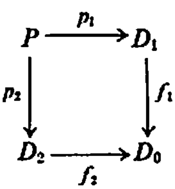
§21 PULLBACKS AND PUSHOUTS

In this section we investigate some important special cases of the general notions of limit and colimit.

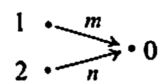
Definitions and Examples

21.1 DEFINITION

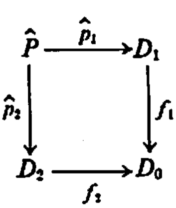
(1) The square in $\mathscr{C}$



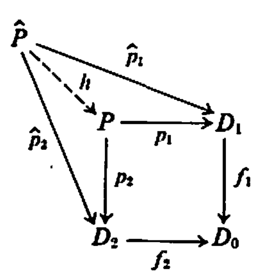
is called a **pullback square** provided that $(P, (p_i)_{i=0,1,2})$ is a limit of the functor $D: I \to \mathscr{C}$ where I is the category



$D(m) = f_1$, $D(n) = f_2$, and $p_0 = f_1 \circ p_1 = f_2 \circ p_2$. In other words, the square is a pullback square if it commutes and for any commutative square of the form:



there exists a unique morphism $h: \hat{P} \to P$ such that the triangles in the diagram



commute.

(2) If

$$\begin{array}{ccc} P & \xrightarrow{p_1} & D_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ D_2 & \xrightarrow[f_2]{} & D_0 \end{array}$$

is a pullback square, then p_2 is said to be a **pullback of f_1 along f_2**. In the case that f_1 is a monomorphism, p_2 is commonly called an **inverse image of f_1 along f_2**.
(3) We say that $\mathscr{C}$ **has pullbacks** provided that each functor from

$$\begin{array}{ccc} \bullet & \searrow & \\ & & \bullet \\ \bullet & \nearrow & \end{array}$$

to $\mathscr{C}$ has a limit; i.e., provided that each figure

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow{\scriptstyle f_1} \\ \bullet & \xrightarrow[f_2]{} & \bullet \end{array}$$

in $\mathscr{C}$ can be extended to become a pullback square. $\mathscr{C}$ **has inverse images** provided that each figure

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow{\scriptstyle f_1} \\ \bullet & \xrightarrow[f_2]{} & \bullet \end{array}$$

in $\mathscr{C}$, with f_1 a monomorphism, can be extended to become a pullback square.

DUAL NOTIONS: **Pushout square; pushout of f_1 along f_2; direct image of f_1 along f_2; has pushouts; has direct images**; in particular, the square

$$\begin{array}{ccc} D_0 & \xrightarrow{f_1} & D_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle p_1} \\ D_2 & \xrightarrow[p_2]{} & P \end{array}$$

is called a **pushout square** provided that $((p_i)_{i=0,1,2}, P)$ is a colimit of $D: I \to \mathscr{C}$, where I is the category

$$\begin{array}{ccc} & \overset{m}{\nearrow} & \bullet\ 1 \\ 0\ \bullet & & \\ & \underset{n}{\searrow} & \bullet\ 2 \end{array}$$

$D(m) = f_1$, $D(n) = f_2$ and $p_0 = p_1 \circ f_1 = p_2 \circ f_2$.

Since limits and colimits are essentially unique, it immediately follows that pullbacks and pushouts are. Thus when they exist, we often speak of *the* pullback (resp. *the* pushout) of a pair of morphisms with common codomain (resp. domain).

21.2 EXAMPLES

(1) If A and B are subsets of the set C, with inclusions $A \hookrightarrow C$ and $B \hookrightarrow C$, then

$$\begin{array}{ccc} A \cap B & \hookrightarrow & A \\ \downarrow & & \downarrow \\ B & \hookrightarrow & C \end{array}$$

is a pullback square in **Set**. Similarly in **Grp** when A and B are subgroups of C and in **Top** when A and B are subspaces of C.

(2) If $f\colon B \to C$ is a function on sets and $A \subset C$, then

$$\begin{array}{ccc} f^{-1}[A] & \hookrightarrow & B \\ {\scriptstyle f|_{f^{-1}[A]}^{A}}\downarrow & & \downarrow{\scriptstyle f} \\ A & \hookrightarrow & C \end{array}$$

is a pullback square in **Set**. [This motivates the terminology "inverse image".]

(3) If $X \xrightarrow{f} B$ and $Y \xrightarrow{g} B$ are morphisms in **Top** and if

$$E = \{(x, y) \mid f(x) = g(y)\} \subset X \times Y$$

has the subspace topology, then

$$\begin{array}{ccc} E & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & B \end{array}$$

is a pullback square in **Top**, where p_X and p_Y are the usual projections restricted to E.†

(4) The construction in (3) "works" in many categories; e.g., **Set**, **Grp**, R-**Mod**, **Rng**, and $\mathbf{BanSp_1}$.

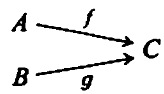
† Notice that if in this example (X, f, B) and (Y, g, B) are considered to be topological bundles with base space B, then $(E, f \circ p_X, B)$ is the fibre product (= Whitney sum) of (X, f, B) and (Y, g, B) and (together with the morphisms p_X and p_Y) is the categorical product of these two objects in $\mathbf{TopBun}_B$ (see Exercise 21C).

Relationship of Pullbacks to Other Limits

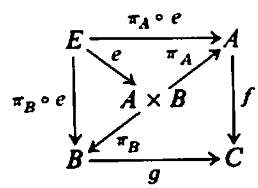
The last example above indicates that in many categories pullbacks can be constructed as equalizers of products. Next we see that this is in fact always true, provided that the corresponding products and equalizers exist.

21.3 THEOREM (CANONICAL CONSTRUCTION OF PULLBACKS)

Let



be a pair of $\mathscr{C}$-morphisms with common codomain. If $(A \times B, \pi_A, \pi_B)$ is a product of (A, B) and if $(E, e) \approx Equ(f \circ \pi_A, g \circ \pi_B)$, then the outer square



is a pullback square.

Proof: The square is constructed so that it commutes. If $q_A: Q \to A$ and $q_B: Q \to B$ such that $f \circ q_A = g \circ q_B$, then by the definition of product, there is a unique morphism $h: Q \to A \times B$ such that $\pi_A \circ h = q_A$ and $\pi_B \circ h = q_B$. Thus $(f \circ \pi_A) \circ h = (g \circ \pi_B) \circ h$, so that since (E, e) is an equalizer of $f \circ \pi_A$ and $g \circ \pi_B$, there is a unique morphism $k: Q \to E$ such that $e \circ k = h$. Hence $(\pi_A \circ e) \circ k = q_A$ and $(\pi_B \circ e) \circ k = q_B$. Also k is unique with respect to this property since products and equalizers are mono-sources. ☐

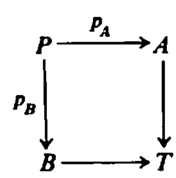
21.4 COROLLARY

If $\mathscr{C}$ has finite products and equalizers, then $\mathscr{C}$ has pullbacks. ☐

21.5 PROPOSITION

If T is a terminal object, then the following are equivalent:

(1)

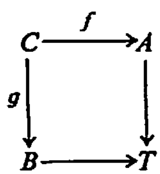


is a pullback square.

(2) *(P, p_A, p_B) is a product of A and B.*

Proof:

(*1*) ⇒ (*2*). Suppose that $f: C \to A$ and $g: C \to B$; then since T is a terminal object, the square



commutes, so that by the definition of pullback, there is a unique morphism $h: C \to P$ such that $f = p_A \circ h$ and $g = p_B \circ h$.

(*2*) ⇒ (*1*). The square commutes, since T is a terminal object. Since (P, p_A, p_B) is a product, any pair of morphisms to A and B can be uniquely factored through p_A and p_B. □

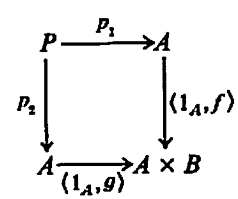
21.6 COROLLARY

If $\mathscr{C}$ has pullbacks and a terminal object, then $\mathscr{C}$ has finite products. □

The condition that $\mathscr{C}$ has a terminal object cannot be deleted in the above corollary, since every non-trivial group (considered as a category) has pullbacks (21B) but no such group has finite products (18K). Also **Field** has pullbacks (21A) but does not have finite products (18L).

21.7 PROPOSITION

If $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are $\mathscr{C}$-morphisms, if $(A \times B, \pi_A, \pi_B)$ exists, and if



is a pullback square, then

(1) $p_1 = p_2$.

(2) $(P, p_1) \approx Equ(\langle 1_A, f\rangle, \langle 1_A, g\rangle)$.

(3) $(P, p_1) \approx Equ(f, g)$.

Proof:

(*1*). Since $\langle 1_A, f\rangle \circ p_1 = \langle 1_A, g\rangle \circ p_2$, we have

$$p_1 = \pi_A \circ \langle 1_A, f\rangle \circ p_1 = \pi_A \circ \langle 1_A, g\rangle \circ p_2 = p_2.$$

(*2*). Immediate from the fact that the square is a pullback square.

(*3*). Suppose that $K \xrightarrow{k} A$ such that $f \circ k = g \circ k$. Then

$$(\pi_B \circ \langle 1_A, f \rangle) \circ k = (\pi_B \circ \langle 1_A \circ g \rangle) \circ k$$

and since $k = k$,

$$(\pi_A \circ \langle 1_A, f \rangle) \circ k = (\pi_A \circ \langle 1_A \circ g \rangle) \circ k.$$

Since the product is a mono-source, we obtain

$$\langle 1_A, f \rangle \circ k = \langle 1_A, g \rangle \circ k$$

so that since the square is a pullback square, there is a unique $h: K \to P$ such that $k = p_1 \circ h$. □

21.8 COROLLARY

If a category $\mathscr{C}$ has finite products and finite intersections, then $\mathscr{C}$ has equalizers. □

21.9 THEOREM

For any category $\mathscr{C}$, the following are equivalent:

(1) *$\mathscr{C}$ has equalizers and finite products.*

(2) *$\mathscr{C}$ has pullbacks and a terminal object.*

Proof:

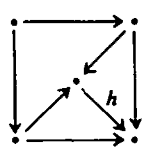
(*1*) ⇒ (*2*). Immediate from Corollary 21.4 and the fact that a terminal object is the product of an empty family.

(*2*) ⇒ (*1*). If $\mathscr{C}$ has pullbacks and a terminal object, then by Corollary 21.6, $\mathscr{C}$ has finite products; hence by Proposition 21.7, $\mathscr{C}$ also has equalizers. □

Relationship of Pullbacks to Special Morphisms

21.10 PROPOSITION

Suppose that the diagram



commutes.

(1) *If the "outer square" is a pullback square, then so is the "inner square".*

(2) *If the "inner square" is a pullback square and h is a monomorphism, then the outer square is a pullback square.* □

21.11 PROPOSITION

If

$$\begin{array}{ccc} P & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback square, then f is a regular epimorphism if and only if $(f, B) \approx Coeq(p_1, p_2)$.

Proof: If f is a regular epimorphism, then $(f, B) \approx Coeq(q_1, q_2)$ for some morphisms $Q \overset{q_1}{\underset{q_2}{\rightrightarrows}} A$. By the definition of pullback square, there exists a morphism $h: Q \to P$ such that $q_1 = p_1 \circ h$ and $q_2 = p_2 \circ h$. Clearly $f \circ p_1 = f \circ p_2$. Now if $g \circ p_1 = g \circ p_2$, then $g \circ p_1 \circ h = g_1 \circ p_2 \circ h$, so that $g \circ q_1 = g \circ q_2$. By the definition of coequalizer, there is a unique morphism k such that $g = k \circ f$. Hence $(f, B) \approx Coeq(p_1, p_2)$. □

21.12 PROPOSITION

In any category $A \xrightarrow{f} B$ is a monomorphism if and only if

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback square. □

21.13 PROPOSITION

Every pullback of:

(1) *a monomorphism is a monomorphism (thus in particular all inverse images are monomorphisms).*

(2) *a regular monomorphism is a regular monomorphism.*

(3) *a retraction is a retraction.*

Proof: Suppose that

$$\begin{array}{ccc} P & \xrightarrow{r} & A \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

is a pullback square.

(*1*). Let f be a monomorphism. If $h, k: Q \to P$ such that $s \circ h = s \circ k$, then

$$f \circ (r \circ h) = g \circ (s \circ h) = g \circ (s \circ k) = f \circ (r \circ k),$$

so that since f is a monomorphism, $r \circ h = r \circ k$. Since pullbacks (being limits) are mono-sources, we can cancel r and s simultaneously, so that $h = k$.

(*2*). Let $(A, f) \approx Equ(p, q)$. Then $(p \circ g) \circ s = (q \circ g) \circ s$. Now if $t: Q \to B$ such that $(p \circ g) \circ t = (q \circ g) \circ t$, then by the definition of equalizer, there is some $u: Q \to A$ such that $f \circ u = g \circ t$. Hence by the definition of pullback, there is some $h: Q \to P$ such that $t = s \circ h$. Moreover, h is unique with respect to this property since by (1), s is a monomorphism. Thus $(P, s) \approx Equ(p \circ g, q \circ g)$.

(*3*). Exercise. □

Congruences

If $f: A \to B$ is a group homomorphism, then the congruence relation determined by f is (in the elementary sense) the subset S of $A \times A$ consisting of all pairs (a, b) with $f(a) = f(b)$. Obviously S can be regarded as a subgroup of $A \times A$, and if $m: S \hookrightarrow A \times A$ is the embedding and $A \times A \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} A$ are the projections, then (according to Theorem 21.3) the square

$$\begin{array}{ccc} S & \xrightarrow{\pi_1 \circ m} & A \\ {\scriptstyle \pi_2 \circ m}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback square.

This motivates our next definition.

21.14 DEFINITION

(1) If

$$\begin{array}{ccc} \bullet & \xrightarrow{p} & \bullet \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[f]{} & \bullet \end{array}$$

is a pullback square, then the pair (p, q) is called a **congruence relation of** f.

(2) A pair (p, q) of $\mathscr{C}$-morphisms is called a **congruence relation** provided that there exists some $\mathscr{C}$-morphism f such that (p, q) is a congruence relation of f.

21.15 PROPOSITION

Let (p, q) be a congruence relation of f. Then

(1) *(p, q) is a congruence relation of $m \circ f$, for each monomorphism m (for which the composition is defined).*

(2) *if* $f = g \circ h$ *and* $h \circ p = h \circ q$, *then* (p, q) *is a congruence relation of* h.

(3) $c \approx Coeq(p, q)$ *implies that* (p, q) *is a congruence relation of* c.

Proof: (1) and (2) follow immediately from Proposition 21.10. To show (3) notice that since $c \approx Coeq(p, q)$, there exists a morphism g such that $f = g \circ c$. Apply (2). □

21.16 PROPOSITION

For any square

$$\begin{array}{ccc} C & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

the following are equivalent:

(1) *The square is both a pullback square and a pushout square* (*i.e., a* **pulation square**, *see* 41.6).

(2) (p, q) *is a congruence relation of* f *and* $f \approx Coeq(p, q)$.

(3) (p, q) *is a congruence relation of* f *and* f *is a regular epimorphism.*

(4) (p, q) *is a congruence relation and* $f \approx Coeq(p, q)$.

Proof: That (1) implies (2) and (2) implies (3) is immediate. That (3) implies (4) follows from Proposition 21.11. To see that (4) implies (1) notice first that since (p, q) is a congruence relation, there exists a unique morphism $t: A \to C$ such that $p \circ t = 1_A$ and $q \circ t = 1_A$. To show that the square is a pushout square, suppose that r and s are morphisms such that $r \circ p = s \circ q$. Then

$$r = r \circ 1_A = r \circ p \circ t = s \circ q \circ t = s \circ 1_A = s.$$

Hence $r \circ p = r \circ q$ so that since $f \approx Coeq(p, q)$, there exists a unique morphism h such that $r = h \circ f = s$. □

21.17 PROPOSITION

In any category, the following are equivalent:

(1) f *is a monomorphism.*

(2) $(1, 1)$ *is a congruence relation of* f.

(3) f *has a congruence relation of the form* (p, p).

Proof: That (1) implies (2) follows from Proposition 21.12 and that (2) implies (3) is trivial. To see that (3) implies (1), suppose that r and s are morphisms such that $f \circ r = f \circ s$. Then by the definition of pullback, there is a morphism h such that $r = p \circ h$ and $s = p \circ h$; hence $r = s$. □

EXERCISES

21A. Show that even though the category **Field** does not have finite products, it does have pullbacks. [Using the canonical construction form the pullback in **Rng**.]

21B. Show that every group (considered as a category) has pullbacks.

21C. Generalize the statement in the footnote to Example 21.2(3) by showing that for any category $\mathscr{C}$

$$\begin{array}{ccc} P & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

is a pullback square in $\mathscr{C}$ if and only if $P \xrightarrow{f \cdot p_1 = g \cdot p_2} C$ together with p_1 and p_2 is a product of $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ in the comma category $(\mathscr{C}, C)$ of $\mathscr{C}$ over C (4.19).

21D. *Galois Correspondence Between Pullbacks and Pushouts*
Show that the process of forming pullbacks and pushouts of commutative squares yields a Galois correspondence in the sense that if P is the process of forming the pullback square

$$\begin{array}{ccc} L & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

of the lower corner of a commutative square

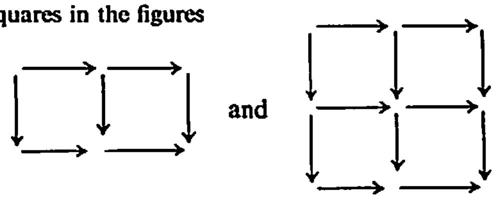
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

and Q is the process of forming the pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & K \end{array}$$

of the upper corner, then $PQP = P$ and $QPQ = Q$.

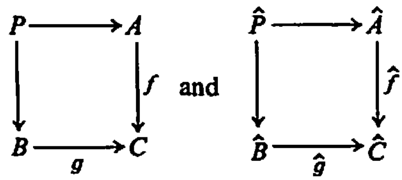
21E. *Pasting and Cancelling Pullback Squares*
Show that:
(a) if the smaller squares in the figures



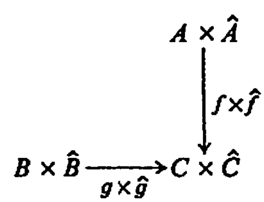
A and B

are pullback squares, then so is the large rectangle and the large square; i.e., pullbacks can be composed by "pasting their edges together".
(b) if figure A commutes and its outer rectangle and right square are pullback squares, then its left square is a pullback square.

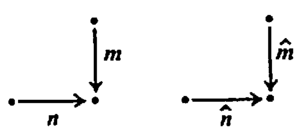
21F. *Pullbacks and Products Commute*
Show, in any category $\mathscr{C}$, that the product of pullbacks is a pullback of the products; specifically if



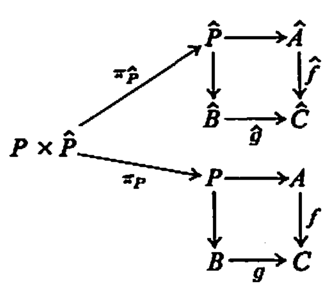
are pullbacks and if $A \times \hat{A}$, $B \times \hat{B}$, and $C \times \hat{C}$ exist, then $P \times \hat{P}$ exists if and only if the pullback $(\bar{P}, p_1, p_2)$ of



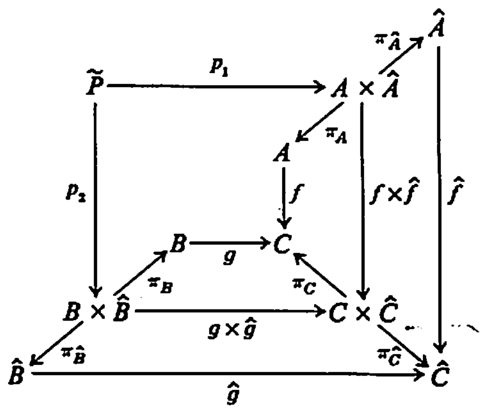
exists, and if they exist, $P \times \hat{P}$ and $\bar{P}$ are isomorphic; moreover, if I is the category



and $D: I \to \mathscr{C}$ is the functor defined by $D(m) = f$, $D(n) = g$, $D(\hat{m}) = \hat{f}$, and $D(\hat{n}) = \hat{g}$, then $P \times \hat{P}$ together with the six morphisms to A, B, C, $\hat{A}$, $\hat{B}$, and $\hat{C}$ indicated by the figure



is a limit of D, and $\tilde{P}$ together with the six morphisms to A, B, C, $\hat{A}$, $\hat{B}$, and $\hat{C}$ indicated by the figure



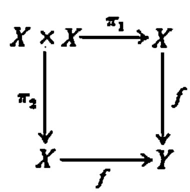
is also a limit of D.

21G. Suppose that the pair of morphisms (f, g) has a coequalizer. Prove that (f, g) is a congruence relation if and only if it is a congruence relation of $Coeq(f, g)$.

21H. Show that if $f: X \to Y$ is a morphism and $(X \times X, \pi_1, \pi_2)$ is the product of X with itself, then the following are equivalent:

(i) f is a constant morphism.

(ii)



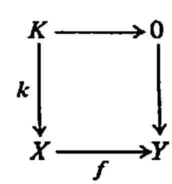
is a pullback square; i.e., (π_1, π_2) is a congruence relation of f.

(iii) $f \circ \pi_1 = f \circ \pi_2$.

21I. Show that if $f: X \to Y$ and $k: K \to X$ are morphisms in a category that has a zero object 0, then the following are equivalent:

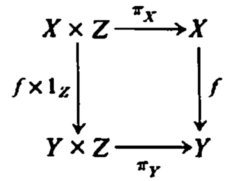
(i) $(K, k) \approx Ker(f)$.

(ii)



is a pullback square.

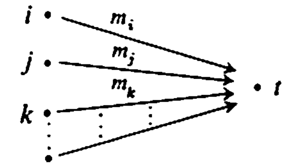
21J. Prove that if $X \xrightarrow{f} Y$ is a morphism and Z is an object in any category, then



is a pullback square.

21K. *Multiple Pullbacks*

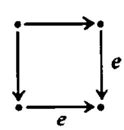
In any category the **multiple pullback** of a sink $((f_i)_I, A)$ is defined to be the limit $(L, (l_i), d)$ (if it exists) of the sink regarded as a functor D in the following way: the indexing set I is considered to be a discrete category and a new category $\hat{I}$ is formed from it by adjoining a terminal object t; i.e., $\hat{I}$ has the form:



and the functor D is defined by $D(m_i) = f_i$, for each $i \in I$. (Notice that for all $i \in I$, $d = f_i \circ l_i$.) We say that a category **has multiple pullbacks** (resp. **has finite multiple pullbacks**) provided that every set-indexed sink (resp. finite sink) has a limit. The dual notion is that of **multiple pushouts**.

(a) Interpret intersections as multiple pullbacks.
(b) Prove that a category has finite multiple pullbacks if and only if it has pullbacks.
(c) State and prove analogues of 21.3, 21.4, 21.5, 21.6, 21.7, 21.8, 21.9, and 21C for multiple pullbacks.

21L. Prove that if $\mathscr{C}$ has pullbacks, then the following are equivalent:
(i) Every epimorphism in $\mathscr{C}$ is a regular epimorphism.
(ii) If e is an epimorphism in $\mathscr{C}$ and



is a pullback square, then it is also a pushout square.

21M. Show that even though the pullback of a retraction is a retraction, the pullback of a section is not necessarily a section (cf. 21.13).

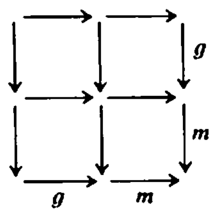
21N. *Pullbacks of Epimorphisms*

(a) Show that in **Set** and **Top** the pullback of an epimorphism is an epimorphism.
(b) Show that in **Grp**, R-**Mod**, **Lat**, **Rng**, and **Mon** the pullback of a regular epimorphism is a regular epimorphism.
(c) Show that in general the pullback of an epimorphism (resp. a regular epimorphism) is not necessarily an epimorphism.

21O. *(Regular Epi, Mono)-Factorizations*
Let $\mathscr{C}$ be a category which has pullbacks and coequalizers. Prove that if any of the following conditions is satisfied, then $\mathscr{C}$ is uniquely (regular epi, mono)-factorizable.

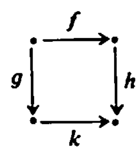
(a) The pullback of each regular epimorphism in $\mathscr{C}$ is an epimorphism.
(b) The class of regular epimorphisms in $\mathscr{C}$ is closed under composition.
(c) There exists a faithful functor $U: \mathscr{C} \to$ **Set** which preserves and reflects regular epimorphisms.

[Let (p, q) be the congruence relation of a $\mathscr{C}$-morphism f, let $g \approx Coeq(p, q)$, and let m be the unique morphism such that $f = m \circ g$. To show that m is a monomorphism, consider:
For (a): The diagram



of successively constructed pullbacks.
For (b): A factorization $m = h \circ \bar{g}$ with $\bar{g}$ a regular epimorphism. Use the fact that $\bar{g} \circ g \approx Coeq(p, q)$ to show that $\bar{g}$ is an isomorphism.]

21P. Show that every commutative square



in $\mathbf{Top}_2$, for which f is a dense embedding, g is perfect, and k is an embedding, must be a pullback square.

§22 INVERSE AND DIRECT LIMITS

We now investigate some additional important particular cases of the notions of limit and colimit.

22.1 DEFINITION

(1) A **downward-directed class** is a partially-ordered class with the property that each pair of elements has a lower bound.

(2) Any functor from a downward-directed class (considered as a category) into a category $\mathscr{C}$ is called an **inverse system in** $\mathscr{C}$.

(3) If I is downward-directed, $D: I \to \mathscr{C}$ is an inverse system in $\mathscr{C}$, and (L, l_i) is the limit of D, then (L, l_i) is sometimes called the **inverse limit of** D.

(4) $\mathscr{C}$ **has inverse limits** provided that for each downward-directed set I, each functor $D: I \to \mathscr{C}$ has a limit.

DUAL NOTIONS: **Upward-directed class; direct system in $\mathscr{C}$; direct limit; has direct limits.**

There is an apparent inconsistency in terminology when "inverse" limits are defined to be particular limits and "direct" limits are particular colimits. The reason for the "switch" is that historically, in certain categories such as **Grp** and **Top**, inverse and direct limits were defined in a manner consistent with Definition 22.1. This situation is similar to some encountered earlier; e.g., the "free product" of groups is really a categorical coproduct and the "Whitney sum" of topological bundles is really a categorical product (18.4).

22.2 EXAMPLES

(1) Categorical inverse limits and direct limits coincide with the classical notions of inverse (or projective) limits and direct (or inductive) limits in the categories **Set**, **Top**, **Grp**, and R-**Mod**. Each of these categories has both inverse and direct limits.

(2) For a given set A, the family of all finite subsets of A together with inclusion functions is a direct system in **Set**, and A together with all of the inclusions into it is the direct limit of the system.

(3) Similar to (2) above, each group is the direct limit of its finitely generated subgroups, each R-module is the direct limit of its finitely generated submodules and each partially-ordered set is the direct limit of its finite subsets.

(4) A Hausdorff space is compactly generated (i.e., is a k-space) if and only if it is the direct limit in **Top** of its compact subspaces.

(5) Every compact Hausdorff space is an inverse limit of compact metric spaces, and every compact metric space is an inverse limit of polyhedra. However, not every compact Hausdorff space is an inverse limit of polyhedra. The reason for this lies in the fact that inverse systems cannot be composed (or iterated).

(6) In each of the concrete categories **Set**, **SGrp**, **Mon**, **Grp**, R-**Mod**, **Rng**, and **Field**, direct limits can be constructed in the following canonical way: Let I be an upward-directed set and let $D: I \to \mathscr{C}$ be a direct system in $\mathscr{C}$, where for each $i \in Ob(I)$, $D(i) = A_i$ and if $i \leq j$, then

$$D(i \to j) = A_i \xrightarrow{f_{ij}} A_j.$$

Let $U: \mathscr{C} \to$ **Set** be the appropriate forgetful functor and let (μ_i, C) be the coproduct (i.e., disjoint union) of $(U(A_i))$ in **Set**. Define an equivalence relation $\sim$ on C by:

> "if x and y are members of C with $x \in U(A_i)$ and $y \in U(A_j)$, then $x \sim y$ if and only if there is some $k \geq i, j$ such that $U(f_{ik})(x) = U(f_{jk})(y)$."

Let $\natural: C \to C/\sim$ be the corresponding natural quotient map. Then there exists a unique "$\mathscr{C}$-object structure" on $C/\sim$ such that all of the functions

$$\natural \circ \mu_i: U(A_i) \to C/\sim$$

become $\mathscr{C}$-morphisms $l_i: A_i \to \bar{C}$. $(l_i, \bar{C})$ is the direct limit of the given direct family. [Observe in particular that **Field** has direct limits but not coproducts.]

(7) The direct limits in **Top** are constructed in essentially the same manner as that described in (6) except that the topology on $C/\sim$ is taken to be the finest one such that all of the functions $\natural \circ \mu_i$ are continuous.

22.3 PROPOSITION

Let I be a downward-directed class and let $J \subset I$ be an "initial" subclass of I (i.e., for each i in I there is some j in J such that $j \leq i$). If $D: I \to \mathscr{C}$ is a functor, $E: J \to \mathscr{C}$ is the restriction of D to J, and for each $i \in J$ there is some $l_i: L \to D(i)$, then the following are equivalent:

(1) *$(L, (l_i)_J)$ is a limit of E.*

(2) *For each i in $I - J$, there is some $l_i: L \to D(i)$ such that $(L, (l_i)_I)$ is a limit of D.*

[Thus inverse limits are determined by inverse limits of "initial" subsystems. Dually, direct limits are determined by direct limits of "final" subsystems.]

Proof:

(1) $\Rightarrow$ *(2)*. Let i be an object of $I - J$. Then there is some j in J and $m: j \to i$. Let $l_i = D(m) \circ l_j$. Since I is downward-directed, J is initial in I, and $(L, (l_i)_J)$ is a limit of E, this assignment yields a well-defined function and $(L, (l_i)_I)$ is a limit of D.

(2) $\Rightarrow$ *(1)*. Clearly $(L, (l_i)_J)$ is a natural source for E. If $(R, (r_i)_J)$ is also a natural source for E, then, as above, the family $(r_i)_J$ can be augmented to provide a natural source $(R, (r_i)_I)$ for D. This guarantees the existence of a unique $h: R \to L$ such that the appropriate triangles commute. □

22.4 COROLLARY

If I is a downward-directed set with a smallest element i_0, then each functor $D: I \to \mathscr{C}$ has a limit (L, l_i) with $L = D(i_0)$. □

22.5 PROPOSITION

Products are inverse limits of finite subproducts. Specifically, suppose that

(*i*) *$(X_i)_I$ is a family of $\mathscr{C}$-objects;*

(*ii*) *for each finite set $J \subset I$, $(\prod_J X_i, \pi_i)$ is the product of $(X_i)_J$;*

(*iii*) *for all finite subsets J and K such that $J \subset K$, $p_{KJ}: \prod_K X_i \to \prod_J X_i$ is the unique morphism induced by the projections;*

(*iv*) *for each $i \in I$, $h_i: \prod_{\{i\}} X_j \to X_i$ is the projection (iso)morphism; and*

(v) $(L, (l_J))$ *is a natural source for the inverse limit system D induced by the finite products and the morphisms* (p_{KJ}) *between them.*

Then the following are equivalent:

(1) $(L, (l_J))$ *is the limit of D.*

(2) $(L, (h_i \circ l_{\{i\}}))$ *is the product of* $(X_i)_I$.

Proof:

(1) ⇒ *(2).* Clearly $h_i \circ l_{\{i\}}: L \to X_i$. Suppose that for each $i \in I$, $f_i: R \to X_i$. By the definition of product, for each finite set $K \subset I$, there is a unique morphism $r_K: R \to \prod_K X_i$ such that for each $j \in K$,

$$\pi_j \circ r_K = h_j \circ p_{K\{j\}} \circ r_K = f_j.$$

Now if $J \subset K$, then for each $j \in J$

$$\pi_j \circ p_{KJ} \circ r_K = h_j \circ p_{J\{j\}} \circ p_{KJ} \circ r_K = f_j = \pi_j \circ r_J.$$

Hence since products are mono-sources, $p_{KJ} \circ r_K = r_J$. Thus $(R, (r_J))$ is a natural source for the inverse system D, so there is a unique morphism $g: R \to L$ such that for each finite J, $l_J \circ g = r_J$. Thus, in particular for each $i \in I$,

$$h_i \circ l_{\{i\}} \circ g = h_i \circ r_{\{i\}} = f_i$$

and g is easily seen to be unique with respect to this property. Hence $(L, (h_i \circ l_{\{i\}}))$ is the product of $(X_i)_I$.

(2) ⇒ *(1).* If $(R, (r_J))$ is a natural source for the inverse system D, then by the definition of product, there is a unique morphism $g: R \to L$ such that for each $i \in I$

$$h_i \circ l_{\{i\}} \circ g = h_i \circ r_{\{i\}}.$$

Hence for each finite set $J \subset I$, if $i \in J$, then

$$\pi_i \circ r_J = h_i \circ p_{J\{i\}} \circ r_J = h_i \circ r_{\{i\}} = h_i \circ l_{\{i\}} \circ g = \pi_i \circ l_J \circ g.$$

Thus since products are mono-sources, $r_J = l_J \circ g$. □

22.6 COROLLARY

If $\mathscr{C}$ *has finite products and inverse limits, then* $\mathscr{C}$ *has products.* □

EXERCISES

22A. Formulate definitions for inverse and direct limits without using the notions of functor or general limits and colimits.

22B. Prove that a topological space is the direct limit in **Top** of its finite subspaces if and only if it is a finitely-generated space (see 14C).

22C. Show that an abelian group is torsion free if and only if it is a direct limit of free abelian groups.

22D. *Limits and Initial Subcategories* (see correction p. 382)

(a) Let I be any category which has pullbacks, and suppose that J is a full initial subcategory of I (i.e., for each $i \in Ob(I)$, there is some $j \in Ob(J)$ and some $m: j \to i$). If $D: I \to \mathscr{C}$ is a functor, $E: J \to \mathscr{C}$ is the restriction of D to J, and for each $i \in Ob(J)$, there is some $l_i: L \to D(i)$, then show that the following are equivalent:

(1) $(L, (l_i)_J)$ is a limit of E.

(2) For each i in $Ob(I) - Ob(J)$, there is some $l_i: L \to D(i)$ such that $(L, (l_i)_I)$ is a limit of D.

(b) Obtain Proposition 22.3 as an immediate corollary.

22E. *Finitary Functors, Objects, and Categories*

A functor $F: \mathscr{C} \to \mathscr{D}$ is called **finitary** (resp. **strongly finitary**) provided that for each direct limit (l_i, L) of a direct system D in $\mathscr{C}$, $(F(l_i), F(L))$ is an epi-sink (resp. direct limit of $F \circ D$) in $\mathscr{D}$. A $\mathscr{C}$-object A is called (strongly) finitary if and only if the functor $hom(A, \_)$: $\mathscr{C} \to$ **Set** is (strongly) finitary. A concrete category $(\mathscr{C}, U)$ is called **(strongly) finitary** provided that U is (strongly) finitary.

(a) Prove that for any set X the following conditions are equivalent:

(i) X is finite.
(ii) X is a finitary object of **Set**.
(iii) X is a strongly finitary object of **Set**.

(b) Let X be any R-module. Prove that the following conditions are equivalent:

(i) X is finitely presented (i.e., there exists an exact sequence

$$R^m \to R^n \to X \to 0).$$

(ii) X is a finitary object of R-**Mod**.
(iii) X is a strongly finitary object of R-**Mod**.

(c) Let X be a topological space. Prove that the following conditions are equivalent:

(i) X is discrete and compact.
(ii) X is a finitary object of **Top**.

(d) Show that the concrete category of Hausdorff spaces is finitary but not strongly finitary.

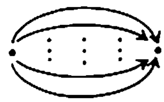
(e) Determine which of the concrete categories given in 2.2 is (strongly) finitary.

§23 COMPLETE CATEGORIES

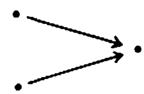
Until now we have investigated limits of functors whose domains are a few special categories. For example, the empty category (which yields terminal objects); the category

$$\bullet \rightrightarrows \bullet$$

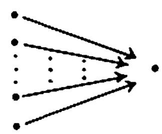
(which yields equalizers); categories of the form



(which yield multiple equalizers); discrete categories (which yield products); the category



(which yields pullbacks); categories of the form



(which yield multiple pullbacks); and downward directed sets (which yield inverse limits).

We have also seen that certain categories such as **Set**, **Grp**, and **Top** have all of these special kinds of limits; i.e., if $\mathscr{C}$ is **Set**, **Grp**, or **Top**, then for each small category I of one of the above types, each functor $D: I \to \mathscr{C}$ has a limit. At this point, because of the myriad of possibilities for small categories, it might seem to be an impossible task to verify that **Set**, **Grp**, or **Top** has a limit for *every* functor from *any* small category. However, we shall see in this section that merely knowing that a category has only a few kinds of limits (e.g., products and equalizers) is enough to guarantee the existence of all limits of small functors. We shall also see that, surprisingly enough, for many categories the property of "having all small limits" is equivalent to the property of "having all small colimits".

Definitions and Preliminaries

23.1 DEFINITION

(1) If I is a category, then the category $\mathscr{C}$ is said to be **I-complete** (or to **have I-limits**) provided that every functor $D: I \to \mathscr{C}$ has a limit.

(2) $\mathscr{C}$ is said to be **complete** provided that $\mathscr{C}$ is I-complete for each small category I.

(3) $\mathscr{C}$ is said to be **finitely complete** (or to **have finite limits**) provided that $\mathscr{C}$ is I-complete for each finite category I.

DUAL NOTIONS: **I^{op}-cocomplete** (or **has I^{op}-colimits**); **cocomplete**; **finitely cocomplete.**

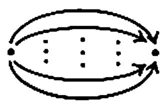
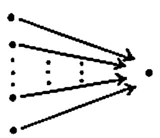
23.2 EXAMPLES

(1) All categories are **1**-complete, **2**-complete and **3**-complete.

(2)

I	$\mathscr{C}$ is I-complete means that $\mathscr{C}$:	$\mathscr{C}$ is I^{op}-cocomplete means that $\mathscr{C}$:
$\varnothing$	has a terminal object	has an initial object
$\bullet \rightrightarrows \bullet$	has equalizers	has coequalizers
$\bullet \quad \bullet$	has products of pairs	has coproducts of pairs
$\bullet \rightarrow \bullet \leftarrow \bullet$	has pullbacks	has pushouts

(3)

I	$\mathscr{C}$ is I-complete for all categories I of this form means that $\mathscr{C}$:	$\mathscr{C}$ is I^{op}-cocomplete for all categories I of this form means that $\mathscr{C}$:
	has multiple equalizers	has multiple coequalizers
small discrete	has products	has coproducts
	has multiple pullbacks	has multiple pushouts
downward directed set	has inverse limits	has direct limits

23.3 PROPOSITION

Let I be a category, $\mathscr{C}$ an I-complete category, and for each functor $D: I \to \mathscr{C}$, let $(L_D, l_i(D))$ be the limit of D. Then there exists a unique functor $\underset{I}{\mathrm{Lim}}: \mathscr{C}^I \to \mathscr{C}$ such that:

(1) *for each $\mathscr{C}^I$-object D, $\underset{I}{\mathrm{Lim}}(D) = L_D$, and*

(2) *for each $\mathscr{C}^I$-morphism $\eta = (\eta_i): D \to E$, the squares*

$$\begin{array}{ccc} \underset{I}{\mathrm{Lim}}(D) & \xrightarrow{l_i(D)} & D(i) \\ {\scriptstyle \underset{I}{\mathrm{Lim}}(\eta)}\downarrow & & \downarrow{\scriptstyle \eta_i} \\ \underset{I}{\mathrm{Lim}}(E) & \xrightarrow[l_i(E)]{} & E(i) \end{array}$$

commute, for each $i \in Ob(I)$.

Proof: $(L_D, \eta_i \circ l_i(D))$ is easily seen to be a natural source for E. Hence there exists a unique morphism $\underset{I}{Lim}(\eta)$ such that the above squares commute for all i. A straightforward verification shows that this defines the required functor. ☐

23.4 DEFINITION

If $\mathscr{C}$ is I-complete, then the functor $\underset{I}{Lim}: \mathscr{C}^I \to \mathscr{C}$ constructed in the above proposition is called the **I-limit functor for $\mathscr{C}$**. Analogously, if $\mathscr{C}$ is I-cocomplete, there exists an **I-colimit functor for $\mathscr{C}$**, denoted by $\underset{I}{Colim}: \mathscr{C}^I \to \mathscr{C}$.

23.5 DEFINITION

(1) A subcategory $\mathscr{A}$ of a category $\mathscr{C}$ (with embedding functor E) is said to be **closed under the formation of I-limits in $\mathscr{C}$** provided that for each functor

$$D: I \to \mathscr{A},$$

each $\mathscr{C}$-limit object of $E \circ D$ lies in $\mathscr{A}$.

(2) A subcategory $\mathscr{A}$ of $\mathscr{C}$ that is closed under the formation of I-limits, for each small category I, is called a **complete subcategory of $\mathscr{C}$**.

DUAL NOTIONS: **closed under the formation of I^{op}-colimits in $\mathscr{C}$; cocomplete subcategory of $\mathscr{C}$.**

If $\mathscr{A}$ is a complete subcategory of $\mathscr{C}$ (with embedding E), $D: I \to \mathscr{A}$, and (L, l_i) is a limit of $E \circ D$, then (L, l_i) is not necessarily a limit of D. (Why not?) However, if $\mathscr{A}$ is full in $\mathscr{C}$, we have the following:

23.6 PROPOSITION

A full complete subcategory of a complete category is itself complete. ☐

It should be pointed out that the converse of the above proposition does not hold; i.e., if $\mathscr{A}$ is a complete category that is also a full subcategory of a complete category $\mathscr{C}$, then $\mathscr{A}$ is not necessarily a complete subcategory of $\mathscr{C}$. Take, for example, $\mathscr{A}$ as the category of abelian torsion groups and $\mathscr{C}$ as **Ab** (Example 23.9(8)) or $\mathscr{A}$ as the category of locally connected topological spaces and $\mathscr{C}$ as **Top** (Example 23.9(7)).

Characterization of Completeness

23.7 THEOREM

For any category, $\mathscr{C}$, the following are equivalent:

(1) *$\mathscr{C}$ is finitely complete.*

(2) *$\mathscr{C}$ has pullbacks and a terminal object.*

(3) *$\mathscr{C}$ has finite products and pullbacks.*

(4) *$\mathscr{C}$ has finite products and inverse images.*

(5) *$\mathscr{C}$ has finite products and finite intersections.*

(6) *$\mathscr{C}$ has finite products and equalizers.*

(7) *$\mathscr{C}$ has finite products, equalizers, and finite intersections of regular subobjects.*

Proof:

(*1*) ⇒ (*2*). Pullbacks and terminal objects are limits of particular functors with finite domains.

(*2*) ⇒ (*3*). Corollary 21.6.

(*3*) ⇒ (*4*). Inverse images are particular pullbacks.

(*4*) ⇒ (*5*). Finite intersections are particular inverse images.

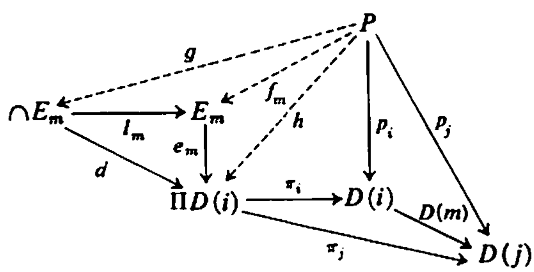
(*5*) ⇒ (*6*). Corollary 21.8.

(*6*) ⇒ (*7*). Corollary 18.25.

(*7*) ⇒ (*1*). Suppose that I is a finite category and $D: I \to \mathscr{C}$. Form the product $(\Pi D(i), \pi_i)$ of the family $(D(i))_{i \in Ob(I)}$ of $\mathscr{C}$-objects. For each I-morphism $i \xrightarrow{m} j$, let

$$(E_m, e_m) \approx Equ(D(m) \circ \pi_i, \pi_j).$$

By hypothesis, the intersection $(\cap E_m, d)$ of the family $(E_m, e_m)_{m \in Mor(I)}$ exists. We claim that $(\cap E_m, (\pi_i \circ d))$ is a limit of D.



Clearly, for each $i \xrightarrow{m} j$, $D(m) \circ \pi_i \circ d = \pi_j \circ d$. Now suppose that $(P, (p_i))$ is also a natural source for D. Then by the definition of product, there is a morphism $h: P \to \Pi D(i)$ such that for each i, $\pi_i \circ h = p_i$. Since $(P, (p_i))$ is a natural source for D, we have that for each $i \xrightarrow{m} j$,

$$D(m) \circ \pi_i \circ h = \pi_j \circ h.$$

Hence by the definition of equalizer, for each such m there is a morphism $f_m: P \to E_m$ such that $e_m \circ f_m = h$. Thus by the definition of intersection, there is a unique morphism $g: P \to \cap E_m$ such that $d \circ g = h$. Consequently, for each i

$$(\pi_i \circ d) \circ g = \qquad \pi_i \circ h = p_i$$

and since products are mono-sources, and d is a monomorphism, g is unique with respect to this property. Consequently $(\cap E_m, (\pi_i \circ d))$ is a limit of D. □

23.8 THEOREM

For any category, $\mathscr{C}$, the following are equivalent:

(1) *$\mathscr{C}$ is complete.*

(2) *$\mathscr{C}$ has multiple pullbacks and a terminal object.*

(3) *$\mathscr{C}$ has products and pullbacks.*

(4) *$\mathscr{C}$ has products and inverse images.*

(5) *$\mathscr{C}$ has products and finite intersections.*

(6) *$\mathscr{C}$ has products and equalizers.*

(7) *$\mathscr{C}$ has products, equalizers, and intersections of regular subobjects.*

(8) *$\mathscr{C}$ is finitely complete and has inverse limits.*

Proof: The equivalence of (1) through (7) can be shown with proofs analogous to those given in the finite case above. This is left as an exercise. Clearly (1) implies (8), since inverse limits are particular limits. That (8) implies (6) follows from the fact that products are inverse limits of finite subproducts (22.6). □

As expected, the above characterization theorems are indispensable when it comes to establishing the completeness and cocompleteness properties of particular categories.

23.9 EXAMPLES

(1) The category of finite sets and the category of finite topological spaces are both finitely complete and finitely cocomplete, but neither is complete or cocomplete.

(2) The category of finite groups is finitely complete, but is not finitely cocomplete.

(3) The categories **Set**, **Grp**, R-**Mod**, and **Top** are complete and cocomplete.

(4) The category of non-empty sets is neither finitely complete nor finitely cocomplete.

(5) The category **Field** is neither finitely complete nor finitely cocomplete.

(6) Each of the categories $\mathbf{BanSp}_1$ and $\mathbf{BanSp}_2$ is finitely complete but not complete.

(7) The category of locally connected topological spaces is complete and cocomplete [however, products and equalizers are not the topological products and subspaces (18.8(5) and 16.3(3))].

(8) The category of abelian torsion groups is complete and cocomplete [products, however, are not the direct products (18.8(4))].

(9) The category of torsion free abelian groups is complete and cocomplete [coequalizers, however, are not the quotient groups (16E)].

(10) A partially-ordered set considered as a category is finitely complete and finitely cocomplete if and only if it is a lattice with a smallest member and a largest member.

(11) If $\mathscr{C}$ is a small category, then the following are equivalent:

(i) $\mathscr{C}$ is complete.

(ii) $\mathscr{C}$ is cocomplete.

(iii) $\mathscr{C}$ is equivalent to a complete lattice (see 18.22).

Completeness Almost Means Cocompleteness

Since completeness and cocompleteness are dual notions, it might seem reasonable to assume that the presence of either condition in a category would have nothing to do with the presence of the other. It is somewhat surprising then that (as the last example above shows) for small categories the two conditions are equivalent. We have previously had another hint that completeness and cocompleteness are in some way related; namely that a terminal object in a category is both the *limit* of the empty functor and the *colimit* of the identity functor (20.3(4) and 20.3(5)). Presently, we shall see that even in quite general settings completeness often implies cocompleteness. That this is not always true is shown by the following three examples. In each example, the category $\mathscr{C}$ is complete but not cocomplete.

23.10 EXAMPLES

(1) Let $\mathscr{C}$ be the opposite of the partially-ordered class of ordinals, considered as a category. $\mathscr{C}$ has no initial object; i.e., no empty coproduct.

(2) Let $\mathscr{C}$ be the category of complete lattices and complete lattice homomorphisms. If X is the three-element lattice, then there is no coproduct of three copies of X. [Hales, 1964.]

(3) Let $\mathscr{C}$ be the category of complete boolean algebras and complete boolean homomorphisms. If X is the four-element complete boolean algebra, then there is no coproduct of countably many copies of X. [Gaifman, 1964; Hales, 1964.]

The following theorems show that under suitable smallness conditions, completeness in a category implies cocompleteness. This is one of the many instances that illuminate the fact that category theory is essentially a "two-pronged" subject, consisting of:

(1) general constructions; and

(2) smallness considerations; which guarantee that the constructions can be carried out within certain "regions" (usually fixed categories).

23.11 THEOREM

Every complete, well-powered, extremally co-(well-powered) category $\mathscr{C}$ has coequalizers.

Proof: Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be given $\mathscr{C}$-morphisms. Let $\mathscr{E}$ be the class of all extremal quotient objects (q, X) of B for which $q \circ f = q \circ g$. Since $\mathscr{C}$ is extremally co-(well-powered), there is a representative class $(h_i, X_i)_I$ of $\mathscr{E}$ that is a set. Since $\mathscr{C}$ is complete, we can form the product $(\Pi X_i, \pi_i)$, the definition of which guarantees the existence of a morphism

$$h = \langle h_i \rangle : B \to \Pi X_i$$

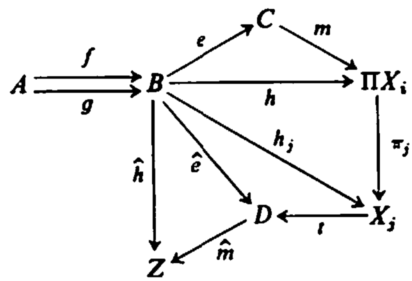
such that for each i, $h_i = \pi_i \circ h$. Since $\mathscr{C}$ is (extremal epi, mono)-factorizable (17.16), there is a factorization of h

$$B \xrightarrow{h} \Pi X_i = B \xrightarrow{e} C \xrightarrow{m} \Pi X_i,$$

where e is an extremal epimorphism and m is a monomorphism. We claim that (e, C) is a coequalizer of f and g. Since for each i,

$$\pi_i \circ m \circ e \circ f = \pi_i \circ h \circ f = h_i \circ f = h_i \circ g = \pi_i \circ h \circ g = \pi_i \circ m \circ e \circ g$$

and since the product is a mono-source and m is a monomorphism, we have that $e \circ f = e \circ g$. Now suppose that $\hat{h}$ is a morphism such that $\hat{h} \circ f = \hat{h} \circ g$. Again by Proposition 17.16, there exists an extremal epimorphism $\hat{e}$ and a monomorphism $\hat{m}$, such that $\hat{h} = \hat{m} \circ \hat{e}$. Since $\hat{m}$ is a monomorphism, $\hat{e} \circ f = \hat{e} \circ g$, so that by the definition of isomorphic quotient objects, there is some $j \in I$ and some isomorphism t such that $t \circ h_j = \hat{e}$.



Thus $r = \hat{m} \circ t \circ \pi_j \circ m$ is a morphism such that $r \circ e = \hat{h}$, and since e is an epimorphism, r is unique. Hence $(e, C) \approx Coeq(f, g)$. □

23.12 COROLLARY

If $\mathscr{C}$ is complete, well-powered, and co-(well-powered), then the following are equivalent:

(1) *$\mathscr{C}$ is cocomplete.*

(2) *$\mathscr{C}$ has coproducts.*

(3) *$\mathscr{C}$ has finite coproducts and direct limits.* □

A complete, well-powered, and co-(well-powered) category does not have to be cocomplete (see Exercise 23D). However, if the category is also strongly co-(well-powered) (19.9), then, as the next theorem shows, it must also be cocomplete.

23.13 THEOREM

If $\mathscr{C}$ is complete and well-powered, then the following are equivalent:

(1) *$\mathscr{C}$ is strongly co-(well-powered).*

(2) *$\mathscr{C}$ is cocomplete and co-(well-powered).*

Proof: That (2) implies (1) is immediate from the dual of Proposition 19.12. To show that (1) implies (2), it suffices (due to the dual of the characterization theorem (23.8)) to show that each complete, well-powered, strongly co-(well-powered) category has coproducts and coequalizers. The proof for coequalizers has been given above (23.11), and since the proof for coproducts is quite similar, we will only provide a sketch of it. If $(A_i)_I$ is a set-indexed family of $\mathscr{C}$-objects, let $(X_k)_K$ be a representative set of $\mathscr{C}$-objects with the property that for each k, there is an epi-sink

$$(A_i \xrightarrow{f_k^i} X_k, X_k).$$

Since (X_k) is a set, we can form the product $(\Pi X_k, \pi_k)$. Then there is an induced sink

$$(A_i \xrightarrow{\langle f_k^i \rangle} \Pi X_k, \Pi X_k)$$

which, by Proposition 19.14, admits an [(extremal epi)-sink, mono]-factorization

$$(A_i) \xrightarrow{(\mu_i)} C \xrightarrow{m} \Pi X_k.$$

Then $((\mu_i), C)$ is a coproduct of $(A_i)_I$. □

The similarities in the proofs of the above theorem and Theorem 23.11 are not accidental. The reason for this is that the construction of these colimits are typical "universal map" constructions and colimits will turn out to be particular "universal maps" (see §26).

The following theorem shows that "having a coseparator" is (for complete, well-powered categories) a very strong smallness condition.

23.14 THEOREM

If $\mathscr{C}$ is complete, well-powered and has a coseparator C, then $\mathscr{C}$ is co-(well-powered) and cocomplete.

Proof: According to the above theorem, it is sufficient to show that $\mathscr{C}$ is strongly co-(well-powered). Let

$$(X_i \xrightarrow{g_i} X, X)$$

be an epi-sink.

Case I. $(X_i) \neq \varnothing$. By the definition of epi-sink, $f \mapsto (f \circ g_i)$ defines an injective function from $hom(X, C)$ into $\prod_I hom(X_i, C)$. Consequently if $hom(X, C) \neq \varnothing$, then $C^{hom(X,C)}$ is a subobject of $C^{\Pi hom(X_i,C)}$ (see Exercise 18F). If $hom(X, C) = \varnothing$, then $C^{hom(X,C)}$ is a terminal object. Since C is a coseparator, X is a subobject of $C^{hom(X,C)}$ (19.6) and therefore it is either a subobject of $C^{\Pi hom(X_i,C)}$ or of the terminal object of $\mathscr{C}$. Since $\mathscr{C}$ is well-powered, there can be no more than a set of such objects which are pairwise non-isomorphic.

Case II. $(X_i) = \varnothing$. Exercise. □

We conclude this section with several partial results.

23.15 THEOREM

If in a complete, well-powered category $\mathscr{C}$, a copower ${}^I A$ of a $\mathscr{C}$-object A exists, then ${}^K A$ exists for each non-empty subset K of I.

Proof: Let $(\mu_i, {}^I\!A)$ be a coproduct and let

$$(A \xrightarrow{\nu_k} B \xrightarrow{m} {}^I\!A)$$

be an [(extremal epi)-sink, mono]-factorization of the sink

$$((A \xrightarrow{\mu_k} {}^I\!A)_{k \in K}, {}^I\!A).$$

Then (ν_k, B) is a coproduct of $(A)_{k \in K}$. To prove this, let

$$(A \xrightarrow{f_k} C)_{k \in K}$$

be a family of $\mathscr{C}$-morphisms. Fix an arbitrary $k_0 \in K$ and define $f_i = f_{k_0}$ for each $i \in I - K$. Then there exists a morphism $f\colon {}^I\!A \to C$ such that $f_i = f \circ \mu_i$ for each i. Consequently, the morphism $g = f \circ m\colon B \to C$ has the property that $f_k = g \circ \nu_k$ for each $k \in K$. Since $((\nu_k), B)$ is an epi-sink, g is uniquely determined by this property. □

23.16 THEOREM

If $\mathscr{C}$ is complete, well-powered and connected, and $\coprod X_i$ exists for a family (X_i), then $\coprod X_k$ exists for each subfamily (X_k) of (X_i), including the empty subfamily.

Proof: Similar to that given for Theorem 23.15 above. □

23.17 THEOREM

If $\mathscr{C}$ is complete, well-powered, co-(well-powered), and has a separator S, then the following are equivalent:

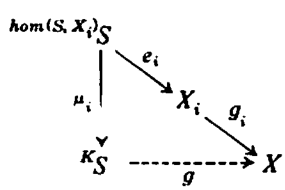
(1) *$\mathscr{C}$ is cocomplete.*

(2) *$\mathscr{C}$ has arbitrary copowers of S.*

Proof: (1) obviously implies (2). To show that (2) implies (1), according to Theorem 23.13, it is sufficient to show that $\mathscr{C}$ is strongly co-(well-powered). Let

$$(X_i \xrightarrow{g_i} X, X)_I$$

be an epi-sink in $\mathscr{C}$. Then by the dual of Proposition 19.6, each X_i is a quotient of ${}^{hom(S,X_i)}S$. Let ${}^{hom(S,X_i)}S \xrightarrow{e_i} X_i$ be the corresponding epimorphisms. Let $K = \coprod_I hom(S, X_i)$, and let $(\mu_i, {}^K S)$ be the coproduct of the family $({}^{hom(S,X_i)}S)_I$. Then there exists a morphism $g\colon {}^K S \to X$ such that the diagram



commutes for each i. g is obviously an epimorphism and X is consequently a quotient of ${}^K S$. Since K does not depend on X and $\mathscr{C}$ is co-(well-powered), the assertion follows. □

EXERCISES

23A. Prove that the category $\mathbf{CompT_2}$ has products and equalizers and conclude that every inverse system of compact Hausdorff spaces has an inverse limit.

23B. *Countably Complete Categories*
A category $\mathscr{C}$ is called **countably complete** provided that $\mathscr{C}$ is I-complete for each category I whose class of morphisms is a countable set.

(a) State and prove a characterization theorem for countably complete categories analogous to the characterization theorems for finite completeness (23.7) and completeness (23.8).

(b) Prove that the category of metrizable topological spaces and continuous functions is countably complete, but is not complete.

23C. *Subcategories Closed Under Certain Constructions I*

(a) For each full subcategory $\mathscr{A}$ of a category $\mathscr{C}$, which has products, let $P\mathscr{A}$ denote the full subcategory of $\mathscr{C}$ whose objects are (the object part of) products of $\mathscr{A}$-objects; and let $S\mathscr{A}$ denote the full subcategory of $\mathscr{C}$ whose objects are (the object part of) subobjects of $\mathscr{A}$-objects. Prove the following:

(i) $PP\mathscr{A} = P\mathscr{A}$.
(ii) $SS\mathscr{A} = S\mathscr{A}$.
(iii) $PS\mathscr{A} \subset SP\mathscr{A}$.
(iv) $SP\mathscr{A}$ is the smallest full subcategory of $\mathscr{C}$ containing $\mathscr{A}$ and closed under the formation of products and subobjects.

(b) Let $\mathscr{A}$ be the full subcategory of **Top** whose objects are the discrete spaces. Determine whether or not $\mathscr{A}$ is a complete category and whether or not it is a complete subcategory of **Top**.

(c) Prove that if $\mathscr{A}$ is a full, isomorphism-closed subcategory of **Ab**, then the following are equivalent:

(i) $\mathscr{A}$ is closed under the formation of products, coproducts, subobjects, and quotient objects.
(ii) There exists a natural number n such that $Ob(\mathscr{A})$ consists of all abelian groups G such that $nG = 0$.

23D. *A Complete, Well-Powered, Co-(Well-Powered) Category Which is Not Cocomplete*
Let $\mathscr{C}$ be the category whose objects are those triples (A, λ, μ) where A is a set and $\lambda: A \to A$ and $\mu: \mathscr{P}(A)-\{\varnothing\} \to A$ are functions (recall that $\mathscr{P}(A)$ denotes the power set of A), where a $\mathscr{C}$-morphism from (A, λ, μ) to $(B, \bar{\lambda}, \bar{\mu})$ is a function $f: A \to B$ such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\lambda} & A & \xleftarrow{\mu} & \mathscr{P}(A)-\{\varnothing\} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f[\ ]} \\ B & \xrightarrow[\bar{\lambda}]{} & B & \xleftarrow[\bar{\mu}]{} & \mathscr{P}(B)-\{\varnothing\} \end{array}$$

commutes.

(a) Verify that $\mathscr{C}$ is indeed a category.

(b) Show that $\mathscr{C}$ has products and equalizers and is thus complete.

(c) Show that the $\mathscr{C}$-monomorphisms are precisely the $\mathscr{C}$-morphisms that are injective

functions, and conclude that $\mathscr{C}$ is well-powered. [*Hint:* If a straightforward proof fails, show that the forgetful functor to Set preserves products and equalizers and so by Proposition 21.12 must preserve monomorphisms.]
(d) Show that the $\mathscr{C}$-epimorphisms are precisely the $\mathscr{C}$-morphisms that are surjective functions, and conclude that $\mathscr{C}$ is co-(well-powered). [*Hint:* Make a construction similar to that used to characterize the epimorphisms in $\mathbf{Top}_2$ (6.10(4)).]
(e) Show that $\mathscr{C}$ is not strongly co-(well-powered), and conclude that it is not cocomplete. [*Hint:* Let X be the $\mathscr{C}$-object (P, λ, μ) where P is a singleton set, and for each ordinal number α, let A_α be the partially-ordered set formed by adjoining two non-comparable minimal elements α_1 and α_2 to the set of all ordinals less than or equal to α. Define $\lambda_\alpha: A_\alpha \to A_\alpha$ and $\mu_\alpha: \mathscr{P}(A_\alpha) - \{\varnothing\} \to A_\alpha$ by:

$$\lambda_\alpha(x) = \begin{cases} \text{the immediate successor of } x & \text{if } x \notin \{\alpha_1, \alpha_2, \alpha\} \\ \alpha_1 & \text{if } x = \alpha_1 \\ \alpha_2 & \text{if } x = \alpha_2 \\ \alpha & \text{if } x = \alpha \end{cases}$$

$$\mu_\alpha(B) = sup(B).$$

Show that there is a proper class of non-isomorphic $\mathscr{C}$-objects Y such that there is an epi-sink from the pair (X, X) to Y.]

§24 FUNCTORS WHICH PRESERVE AND REFLECT LIMITS

In this section we consider the notions of preservation and reflection of limits and colimits. As it turns out, many of the important functors in mathematics (including many of the forgetful and inclusion functors) do preserve limits. In the next chapter we will see that the concept of limit preservation is intimately connected with the very important concepts of universal maps and adjoint situations.

24.1 DEFINITION

Let I be a category and $F: \mathscr{A} \to \mathscr{B}$ be a functor. F is said to

(1) **preserve I-limits** provided that whenever $D: I \to \mathscr{A}$ is a functor and $(L, (l_i))$ is a limit of D, then $(F(L), (F(l_i)))$ is a limit of $F \circ D: I \to \mathscr{B}$.

(2) **reflect I-limits** provided that whenever $D: I \to \mathscr{A}$ is a functor and $(L, (l_i))$ is a source in $\mathscr{A}$ such that $(F(L), (F(l_i)))$ is a limit of $F \circ D$, then $(L, (l_i))$ is a limit of D.

(3) **preserve limits** (resp. **preserve finite limits**) provided that F preserves I-limits for every small category (resp. finite category) I.

(4) **reflect limits** (resp. **reflect finite limits**) provided that F reflects I-limits for every small category (resp. finite category) I.

DUAL NOTIONS: **preserve I^{op}-colimits, reflect I^{op}-colimits, preserve (finite) colimits, reflect (finite) colimits.**

It should be noticed that the above definition is a general one which automatically carries with it many translations to special cases. For example, from the general definition one obtains the definitions:

F **preserves equalizers** *provided that F preserves I-limits, where I is the category* $\bullet \rightrightarrows \bullet$ and

F **preserves products** *provided that F preserves I-limits for each small discrete category I.*

Also from the proofs of the characterization theorems for finite completeness and completeness (23.7 and 23.8), we immediately obtain the following theorems:

24.2 THEOREM

If $\mathscr{A}$ is finitely complete and $F: \mathscr{A} \to \mathscr{B}$ is a functor, then the following are equivalent:

(1) *F preserves finite limits.*

(2) *F preserves pullbacks and terminal objects.*

(3) *F preserves finite products and pullbacks.*

(4) *F preserves finite products and inverse images.*

(5) *F preserves finite products and finite intersections.*

(6) *F preserves finite products and equalizers.* □

24.3 THEOREM

If $\mathscr{A}$ is complete and $F: \mathscr{A} \to \mathscr{B}$ is a functor, then the following are equivalent:

(1) *F preserves limits.*

(2) *F preserves multiple pullbacks and terminal objects.*

(3) *F preserves products and pullbacks.*

(4) *F preserves products and inverse images.*

(5) *F preserves products and finite intersections.*

(6) *F preserves products and equalizers.*

(7) *F preserves finite limits and inverse limits.* □

24.4 EXAMPLES

(1) The forgetful functors from **Grp**, *R*-**Mod**, **SGrp**, **Mon**, **Rng**, **BooAlg**, and **Lat** to **Set** preserve and reflect limits and direct limits, but none of them preserves or reflects arbitrary colimits.

(2) The forgetful functor from **Top** to **Set** preserves limits and colimits but does not reflect either.

(3) The embedding functor from **Ab** to **Grp** preserves limits but not colimits.

(4) The forgetful functor from the category of finite abelian (resp. abelian torsion) groups to **Set** preserves finite limits, but not arbitrary limits (see Exercise 24F).

(5) All covariant *hom*-functors $hom(A, \_)$ preserve limits (see Theorem 29.3).

(6) If $\mathscr{A}$ has (finite) products, then for each $\mathscr{A}$-object, A, the functor

$$(A \times \_): \mathscr{A} \to \mathscr{A}$$

preserves limits.

(7) If $\mathscr{A}$ has equalizers and I is the category $\bullet \rightrightarrows \bullet$, then the equalizer functor $E: \mathscr{A}^I \to \mathscr{A}$ (see Exercise 16C) preserves limits.

(8) Examples (6) and (7) above can be derived as special cases of the general fact that any two varieties of limits commute (see §25). Specifically if $\mathscr{A}$ is I-complete, then the functor $\underset{I}{Lim}: \mathscr{A}^I \to \mathscr{A}$ preserves limits and if $\mathscr{A}$ is I-cocomoplete, the functor $\underset{I}{Colim}: \mathscr{A}^I \to \mathscr{A}$ preserves colimits.

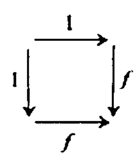
(9) Each "constant functor" functor $C: \mathscr{A} \to \mathscr{A}^I$ (see 15.8) preserves both limits and colimits.

(10) Let $\mathscr{A}$ be a full subcategory of $\mathscr{B}$ with embedding functor $E: \mathscr{A} \hookrightarrow \mathscr{B}$. E reflects both limits and colimits, and if $\mathscr{A}$ is complete, then E preserves limits provided that $\mathscr{A}$ is a complete subcategory of $\mathscr{B}$.

24.5 PROPOSITION

If F preserves pullbacks, then F preserves monomorphisms.

Proof: Recall that f is a monomorphism if and only if



is a pullback square. □

24.6 PROPOSITION

If $\mathscr{A}$ has equalizers and $F: \mathscr{A} \to \mathscr{B}$ preserves equalizers, then F is faithful if and only if it reflects epimorphisms.

Proof: That faithful functors reflect epimorphisms has been shown previously (12.8).

To show the converse, let (E, e) be the equalizer of a pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$. Then

$$(F(E), F(e)) \approx Equ(F(f), F(g)),$$

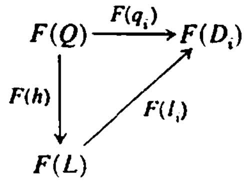
so that if $F(f) = F(g)$, $F(e)$ must be an epimorphism (16.7). Since F reflects epimorphisms, e must be an epimorphism; hence $f = g$ (16.7). □

24.7 THEOREM

If $F: \mathscr{A} \to \mathscr{B}$ is faithful and reflects isomorphisms and $\mathscr{A}$ has I-limits (resp. I-colimits) and F preserves them, then F reflects them.

Proof: Suppose that $D: I \to \mathscr{A}$ and $(Q, (q_i))$ is a source such that $(F(Q), (F(q_i)))$ is a limit of $F \circ D$. Then $(F(Q), (F(q_i)))$ is a natural source for

$F \circ D$, so that since F is faithful (and hence reflects commutative triangles) $(Q, (q_i))$ must be a natural source for D. Since $\mathscr{A}$ is I-complete, D has a limit $(L, (l_i))$ and there is a morphism $h: Q \to L$ such that for each $i \in Ob(I)$, $q_i = l_i \circ h$. Consequently, for each i the triangle



commutes, so that since both $(F(Q), (F(q_i)))$ and $(F(L), (F(l_i)))$ are limits of $F \circ D$, the morphism $F(h)$ must be an isomorphism. Since F reflects isomorphisms, $h: Q \to L$ must be an isomorphism, so that $(Q, (q_i))$ is a limit of D. The proof for colimits is similar. □

24.8 THEOREM

If $\mathscr{A}$ is complete and $F: \mathscr{A} \to \mathscr{B}$ preserves limits and reflects isomorphisms, then F is faithful and reflects limits, monomorphisms, and epimorphisms.

Proof: To show that F is faithful, let $\bullet \overset{r}{\underset{s}{\rightrightarrows}} \bullet$ be $\mathscr{A}$-morphisms with $F(r) = F(s)$. Let $(E, e) \approx Equ(r, s)$; then

$$(F(E), F(e)) \approx Equ(F(r), F(s)),$$

so that since $F(r) = F(s)$, $F(e)$ must be an isomorphism (16.7). Since F reflects isomorphisms, e must be an isomorphism. Thus $r = s$ (16.7).

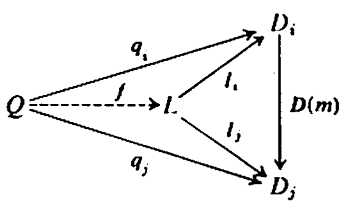
Since F is faithful, it must reflect limits (24.7), monomorphisms, and epimorphisms (12.8). □

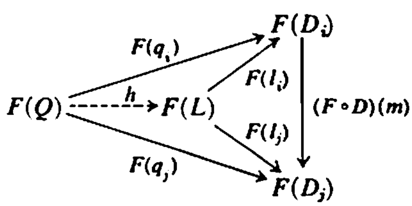
In §12 we have investigated the preservation and reflection properties of functors that are full, faithful, or dense. We conclude this section with a continuation of this study.

24.9 PROPOSITION

Every full faithful functor reflects I-limits and I-colimits for each category I.

Proof: We give a proof for limits. The proof for colimits is dual. Suppose that $I \xrightarrow{D} \mathscr{A} \xrightarrow{F} \mathscr{B}$, F is full and faithful, $(L, (l_i))$ is a source in $\mathscr{A}$, and that $(F(L), (F(l_i)))$ is a limit of $F \circ D$. Since F is faithful, $(L, (l_i))$ must be a natural source for D. Now suppose that $(Q, (q_i))$ is also a natural source for D. Then $(F(Q), (F(q_i)))$ is a natural source for $F \circ D$, so there is a unique morphism $h: F(Q) \to F(L)$ such that for all i, $F(q_i) = F(l_i) \circ h$.





Since F is full, there is some $f: Q \to L$ such that $F(f) = h$ and by the faithfulness of F, f has the property that for all i, $q_i = l_i \circ f$ and is the only morphism with this property. Consequently $(L, (l_i))$ is a limit of D. □

Every full and faithful functor $F: \mathscr{A} \to$ **Set** (where $F(A) \neq \varnothing$ for at least one $A \in Ob(\mathscr{A})$) not only reflects limits, but preserves them as well. Also, if $F(A)$ has at least two elements for some $A \in Ob(\mathscr{A})$, then F preserves and reflects colimits (see Exercise 24D). These are only some of the rather surprising facts about set-valued functors. These functors will be treated more comprehensively in Chapter VIII.

24.10 LEMMA

If F and G are naturally isomorphic functors and I is a category, then F preserves or reflects I-limits (resp. I-colimits) if and only if G does so. □

24.11 PROPOSITION

Each equivalence preserves and reflects both I-limits and I-colimits, for every category I.

Proof: That each equivalence reflects I-limits and I-colimits follows immediately from the fact that it is full and faithful (24.9). We will show limit preservation. The proof for colimits is dual.

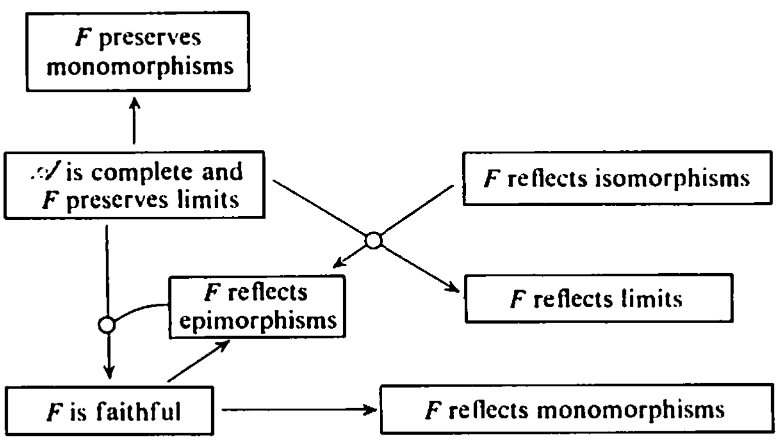
Suppose that $F: \mathscr{A} \to \mathscr{B}$ is an equivalence, $D: I \to \mathscr{A}$, and $(L, (l_i))$ is a limit of D. Since F is an equivalence, there is an equivalence $G: \mathscr{B} \to \mathscr{A}$ such that $G \circ F$ is naturally isomorphic to $1_{\mathscr{A}}$ (14.11). Hence by the lemma,

$$((G \circ F)(L), (G \circ F)(l_i))$$

is a limit of $G \circ F \circ D$. Now G, being an equivalence, is full and faithful; so it reflects limits (24.9). Hence $(F(L), (F(l_i)))$ is a limit of $F \circ D$. □

Notice that the above proposition shows that completeness and cocompleteness are categorical properties.

24.12 Some of the results of this section are summarized in the following figure. (F is assumed to be a functor with domain $\mathscr{A}$.)



EXERCISES

24A. Prove that if $F: \mathcal{A} \to \mathcal{B}$ preserves limits and $\mathcal{A}$ has an initial object P, then F has a limit; namely $(F(P), (F(f_A)))$, where f_A is the unique morphism from P to A.

24B. Construct an example to show that the property: "F preserves coequalizers" is not the same as the property "F preserves regular epimorphisms". [Consider the forgetful functor U: **Grp** → **Set**.]

24C. Prove that if $\mathcal{A}$ is complete and $F: \mathcal{A} \to \mathcal{B}$ preserves limits and reflects isomorphisms, then F reflects extremal epimorphisms.

24D. Suppose that $F: \mathcal{A} \to$ **Set** is full and faithful.
(a) If $F(A) \neq \emptyset$ for some $\mathcal{A}$-object A, prove that F preserves limits.
(b) If $F(A)$ has at least two elements for some $\mathcal{A}$-object A, prove that F preserves colimits.

24E. Suppose that F, G, and H are functors such that $G \circ F = H$. Show that if H preserves limits and G reflects limits, then F preserves limits.

24F. Prove that the forgetful functor from the category of finite abelian groups into **Set** preserves finite limits but not arbitrary limits. [Consider the limit of a functor whose image is $\cdots \to \mathbf{Z}/8\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$.]

24G. Let $\mathcal{A}$ be a full subcategory of $\mathcal{B}$ that contains a separator for $\mathcal{B}$. Prove that the embedding functor $E: \mathcal{A} \hookrightarrow \mathcal{B}$ reflects regular epimorphisms.

§25 LIMITS IN FUNCTOR CATEGORIES

When considering particular limits, we have seen several instances where certain types of limits can be "commuted"; for example, products can be iterated (18.13), products of equalizers are equalizers of products (18.17), pullback squares can be composed by "pasting their edges together" (21E), and products of pullbacks are pullbacks of products (21F). In this section, we will prove the general result that *any* two types of limits commute, obtaining in the process the above results as special cases and also obtaining the consequence that each functor category $\mathcal{B}^{\mathcal{A}}$ inherits the completeness and cocompleteness properties of $\mathcal{B}$. To do this we will begin by showing that if $D: I \times J \to \mathcal{C}$ is a functor and $\mathcal{C}$ is I-complete and J-complete, then the limit of D can be constructed "pointwise", i.e., by first finding the limit of each of the right associated functors $D(i, \_)$, where i is an object of I, and then by finding the limit of the functor "induced" by these limits (or, alternatively, by first finding the limits of each of the functors $D(\_, j)$ and then the limit of the functor "induced" by these limits). We first illustrate the technique with a special case that has been considered before:

25.1 EXAMPLE
Equalizers and products (of pairs) commute.

Let I be the discrete category $\overset{1}{\bullet} \quad \overset{2}{\bullet}$ and let J be the category $i \underset{n}{\overset{m}{\rightrightarrows}} j$.

Then $I \times J$ is

$$(1, i) \underset{(1,n)}{\overset{(1,m)}{\rightrightarrows}} (1, j)$$

$$(2, i) \underset{(2,n)}{\overset{(2,m)}{\rightrightarrows}} (2, j).$$

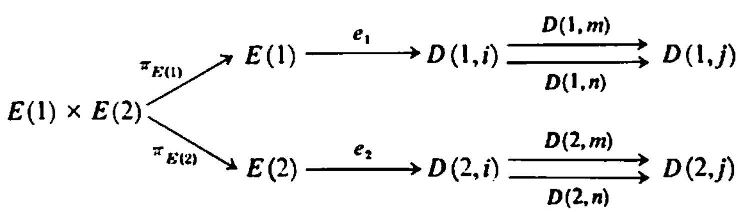
Suppose that $\mathscr{C}$ has products (of pairs) and equalizers, and let $D: I \times J \to \mathscr{C}$ be a functor. Then in $\mathscr{C}$ we have the diagram

$$D(1, i) \underset{D(1,n)}{\overset{D(1,m)}{\rightrightarrows}} D(1, j)$$

$$D(2, i) \underset{D(2,n)}{\overset{D(2,m)}{\rightrightarrows}} D(2, j).$$

Now $D(1, \_\_)$ and $D(2, \_\_)$ are functors from J to $\mathscr{C}$ whose limits are the equalizers of the top and bottom pairs, respectively. Call them $(E(1), e_1)$ and $(E(2), e_2)$. This then defines a functor $E: I \to \mathscr{C}$, and it is easy to verify that the limit of this functor, namely $(E(1) \times E(2), \pi_{E(1)}, \pi_{E(2)})$ is such that when the projection morphisms are composed with the other morphisms given, the result is a limit of D, i.e.,

$$\underset{I \times J}{Lim}\, D \approx (E(1) \times E(2), e_1 \circ \pi_{E(1)}, D(1, m) \circ e_1 \circ \pi_{E(1)}, e_2 \circ \pi_{E(2)}, D(2, m) \circ e_2 \circ \pi_{E(2)}).$$



Similarly, when we consider the functors $D(\_\_, i)$ and $D(\_\_, j)$ from I to $\mathscr{C}$, we see that they have limits which are products. Let

$$F(i) = D(1, i) \times D(2, i)$$

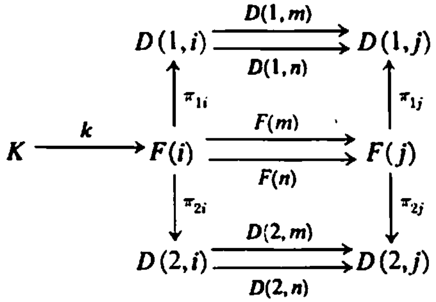
$$F(j) = D(1, j) \times D(2, j)$$

$$F(m) = D(1, m) \times D(2, m)$$

$$F(n) = D(1, n) \times D(2, n).$$

Then we have that F is a functor from J to $\mathscr{C}$ whose limit (namely the equalizer (K, k) of $F(m)$ and $F(n)$) is such that when k is composed with the other morphisms given, a limit of D is obtained; i.e.,

$$\underset{I \times J}{Lim}\, D \approx (K, \pi_{1i} \circ k, D(1, m) \circ \pi_{1i} \circ k, \pi_{2i} \circ k, D(2, m) \circ \pi_{2i} \circ k)$$

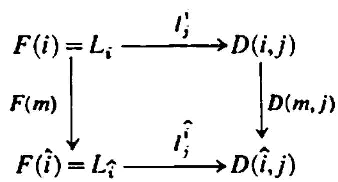


Notice that in the first case we took a product of equalizers to obtain a limit of D, and in the second case we took the equalizer of products. Thus the product of the equalizers of any two pairs of morphisms in $\mathscr{C}$ is essentially no different from the equalizer of the product morphisms formed from the two pairs.

25.2 THEOREM (POINTWISE EVALUATION)

Suppose that I, J, and $\mathscr{C}$ are categories, $D: I \times J \to \mathscr{C}$ is a functor, and for each object i in I, the right associated functor $D(i, \_\_): J \to \mathscr{C}$ has a limit, namely $(L_i, (l_j^i)_J)$. Then

(1) *there is a unique functor $F: I \to \mathscr{C}$ such that for each object i in I, $F(i) = L_i$ and for each morphism $m: i \to \hat{i}$ in I and each object j in J, the diagram*



commutes.

(2) *D has a limit if and only if F has a limit, and any source $(L, (p_i)_I)$ is a limit of F if and only if $(L, (l_j^i \circ p_i)_{I \times J})$ is a limit of D.*

Proof:

(1) For each $m: i \to \hat{i}$ in I, let $F(m)$ be the morphism from L_i to $L_{\hat{i}}$ induced by the fact that $(L_{\hat{i}}, (l_j^{\hat{i}})_J)$ is a limit of $D(\hat{i}, \_\_)$ and $(L_i, (D(m, j) \circ l_j^i)_J)$ is a natural source for $D(\hat{i}, \_\_)$. Since $D(m, \_\_)$ is a natural transformation, F must clearly preserve identities and compositions; hence it is a functor. By the uniqueness condition in the definition of limit, it is evident that F can be defined on morphisms in no other way.

(2) Suppose that $(L, (p_i)_I)$ is a limit of F. Then

$$(L, (l_j^i \circ p_i)_{I \times J})$$

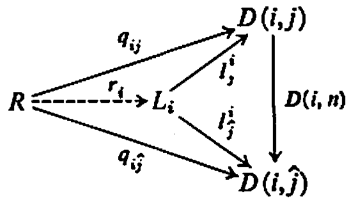
is a natural source for D. Let

$$(R, (q_{ij})_{I \times J})$$

also be a natural source for D. Then for each object i in I and each morphism $j \xrightarrow{n} \hat{j}$ in J,

$$q_{i\hat{j}} = D(i, n) \circ q_{ij}.$$

Hence there is a morphism $r_i \colon R \to L^i$ such that the diagram

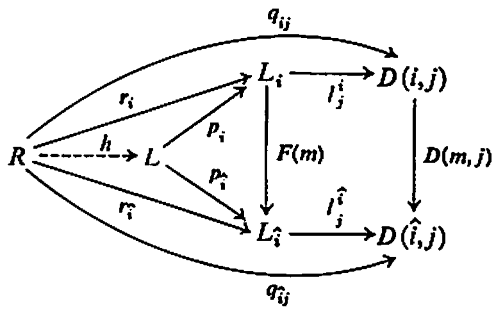


commutes.

Now for each object j in J and each morphism $i \xrightarrow{m} \hat{i}$ in I

$$l_j^{\hat{i}} \circ F(m) \circ r_i = D(m, j) \circ l_j^i \circ r_i = D(m, j) \circ q_{ij} = q_{\hat{i}j} = l_j^{\hat{i}} \circ r_{\hat{i}}$$

so that since $(L_{\hat{i}}, (l_j^{\hat{i}})_J)$ is a mono-source, $F(m) \circ r_i = r_{\hat{i}}$. Hence since $(L, (p_i)_I)$ is a limit of F, there is a unique morphism $h \colon R \to L$ such that the diagram



commutes.

Since each $(L_i, (l_j^i)_J)$ is a mono-source, h is thus unique with respect to the property that for each i and j, $q_{ij} = (l_j^i \circ p_i) \circ h$. Consequently, $(L, (l_j^i \circ p_i)_{I \times J})$ is a limit of D.

The converse follows readily, as does the first part of (2). These are left as exercises. □

25.3 COROLLARY

If $\mathscr{C}$ is I-complete and J-complete, then $\mathscr{C}$ is $I \times J$-complete. □

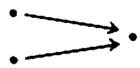
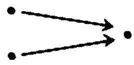
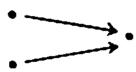
25.4 COROLLARY (COMMUTATION OF LIMITS)

Suppose that $\mathscr{C}$ is I-complete and J-complete and $D \colon I \times J \to \mathscr{C}$ is a functor. Then

$$\underset{I}{Lim}(\underset{J}{Lim}\, D(i, j)) \approx \underset{I \times J}{Lim}\, D \approx \underset{J}{Lim}(\underset{I}{Lim}\, D(i, j)). \quad \square$$

25.5 EXAMPLES

The above result yields the following previously considered special cases:

I	J	Result
discrete	discrete	iteration of products (18.13)
discrete	$\bullet \rightrightarrows \bullet$	products of equalizers are equalizers of products (18.17)
		pullback squares can be composed (21E)
discrete		products of pullbacks are pullbacks of products (21F)

In §15 we have seen that functor categories $\mathscr{B}^{\mathscr{A}}$ often inherit the categorical properties of $\mathscr{B}$. That this is true for completeness and cocompleteness is a consequence of the next theorem.

25.6 THEOREM

Limits in functor categories can be obtained by pointwise evaluation. Specifically, suppose that $D: I \to \mathscr{B}^{\mathscr{A}}$ is a functor and for each $A \in Ob(\mathscr{A})$, $E_A: \mathscr{B}^{\mathscr{A}} \to \mathscr{B}$ is the evaluation functor relative to A (15.8). If for each $A \in Ob(\mathscr{A})$, $E_A \circ D: I \to \mathscr{B}$ has a limit $(L_A, (l_i^A)_I)$, then D has a limit $(F, (q_i)_I)$, where $F: \mathscr{A} \to \mathscr{B}$ is a functor such that $F(A) = L_A$ for each $A \in Ob(\mathscr{A})$, and for each $i \in Ob(I)$, q_i is the natural transformation $(l_i^A)_{A \in Ob(\mathscr{A})}: F \to D(i)$.

Proof: Since the categories $(\mathscr{B}^{\mathscr{A}})^I$ and $\mathscr{B}^{I \times \mathscr{A}}$ are isomorphic, D can be considered as a functor from $I \times \mathscr{A}$ to $\mathscr{B}$ (15.9). Hence, the result is merely a restatement of the Pointwise Evaluation Theorem (25.2). □

25.7 COROLLARY

If $\mathscr{A}$, $\mathscr{B}$, and I are categories, then if $\mathscr{B}$ is I-complete (resp. I-cocomplete), then so is $\mathscr{B}^{\mathscr{A}}$. □

EXERCISES

25A. (a) Use the pointwise evaluation theorem to show that if $\mathscr{B}$ has pullbacks and F and G are objects in $\mathscr{B}^{\mathscr{A}}$, then $\eta: F \to G$ is a monomorphism in $\mathscr{B}^{\mathscr{A}}$ if and only if for each $A \in Ob(\mathscr{A})$, $\eta_A: F(A) \to G(A)$ is a monomorphism in $\mathscr{B}$ (cf. 15.5).

(b) Conclude that if $\mathscr{B}$ has pullbacks and pushouts and is balanced, then $\mathscr{B}^{\mathscr{A}}$ is balanced, for each category $\mathscr{A}$.

25B. *Commutation of limits and colimits*

In this exercise set we are concerned with only the "object parts" of certain limits and colimits; for example, here when we say that in $\mathscr{C}$ "direct limits and finite limits commute" it means that for each upward directed set I, each finite category J, and each functor $D: I \times J \to \mathscr{C}$, the object parts of

$$\underset{I}{Colim}(\underset{J}{Lim}\ D(i,j)) \qquad \text{and} \qquad \underset{J}{Lim}(\underset{I}{Colim}\ D(i,j))$$

are isomorphic.

(a) Show that coproducts and equalizers commute in the categories **Set**, **Top**, and **Cat**.

(b) Show that direct limits and finite limits commute in the categories **Set**, R-**Mod**, **Grp**, **BooAlg**, and **Lat**. [First show that direct limits and pullbacks commute.]

(c) Show that direct limits and (arbitrary) limits do not commute in **Set**. [Let I be the natural numbers $\mathbf{N}$ with the usual order and let J be natural numbers $\mathbf{N}$ considered as a discrete category. Define $D: I \times J \to \mathbf{Set}$ by:

$$D(i,j) = \{n \in \mathbf{N} \mid n \geq i\}$$

and if $f: (i,j) \to (\bar{i},j)$

$$D(f)(n) = sup\{\bar{i}, n\}.]$$