# V

# Functors and Natural Transformations

It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation. . . .

——S. EILENBERG AND S. MAC LANE†

One of our first notions of a category was that of a class of "structured sets" (called objects) together with a class of "structure-preserving functions" (called morphisms) between them. In Chapters III and IV we have seen that generally it is not so much the objects and how they are constructed, as it is the morphisms and how they are composed, that is the focal point of one's attention when investigating categories. In this chapter we step back and take a somewhat wider view—considering categories themselves as structured classes and looking at the "structure-preserving functions" (called functors) between them. Later we will see that, analogously with the earlier situation, much of the importance of the theory of categories lies not in the structure of the categories themselves, but rather in the functors between them and how they are composed. Actually, we can (and do) go one step further by defining and investigating "morphisms" between functors. These are called natural transformations. As expected, the study of natural transformations and how they are composed is the essence of "functor theory".

## §9 FUNCTORS

**9.1 DEFINITION**

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **functor from $\mathscr{C}$ to $\mathscr{D}$** is a triple $(\mathscr{C}, F, \mathscr{D})$ where $F$ is a function from the class of morphisms of $\mathscr{C}$ to the class of morphisms of $\mathscr{D}$ (i.e., $F\colon Mor(\mathscr{C}) \to Mor(\mathscr{D})$) satisfying the following conditions:

† From *Transactions of the American Mathematical Society* 58 (1945).

(1) $F$ **preserves identities**; i.e., if $e$ is a $\mathscr{C}$-identity, then $F(e)$ is a $\mathscr{D}$-identity.

(2) $F$ **preserves composition**; $F(f \circ g) = F(f) \circ F(g)$; i.e., whenever $dom(f) = cod(g)$, then $dom(F(f)) = cod(F(g))$ and the above equality holds.

Instead of writing "$(\mathscr{C}, F, \mathscr{D})$ is a functor", we usually write, more instructively, "$F: \mathscr{C} \to \mathscr{D}$", or $\mathscr{C} \xrightarrow{F} \mathscr{D}$, or "$F$ is a functor from $\mathscr{C}$ to $\mathscr{D}$". Also, as with functions, we usually abuse the notation and write $F$ instead of $(\mathscr{C}, F, \mathscr{D})$. Hence, we write, for example, that $F$ has **domain** $\mathscr{C}$ and **codomain** $\mathscr{D}$. A functor whose domain is a small category is called a **small functor**.

Because for any category there is a one-to-one correspondence $(A \leftrightarrow 1_A)$ between objects and identities (3.2) and because functors preserve identities, each functor $F: \mathscr{C} \to \mathscr{D}$ induces a unique function (also denoted "by abuse of notation" by $F$) from the class of $\mathscr{C}$-objects to the class of $\mathscr{D}$-objects, such that for each $\mathscr{C}$-object $A$

$$F(1_A) = 1_{F(A)}.$$

An immediate consequence is that for all $\mathscr{C}$-objects $A$ and $B$,

$$F[hom_{\mathscr{C}}(A, B)] \subset hom_{\mathscr{D}}(F(A), F(B)).$$

Obviously, then, any functor $F: \mathscr{C} \to \mathscr{D}$ can easily be recovered from its "object-function" $F: Ob(\mathscr{C}) \to Ob(\mathscr{D})$ and all of the restrictions†

$$F\,|_{hom(A,B)}^{hom(FA,FB)}.$$

Because of this, we shall often describe functors by means of their object functions and their "hom-set" restrictions.

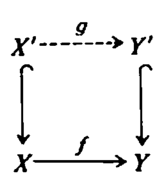
## 9.2 EXAMPLES OF FUNCTORS

(1) For any category $\mathscr{C}$, $(\mathscr{C}, 1_{Mor\mathscr{C}}, \mathscr{C})$ is a functor, called the **identity functor on** $\mathscr{C}$ and denoted by $1_{\mathscr{C}}$.

(2) If $\mathscr{C}$ is a subcategory of $\mathscr{D}$ and $E: Mor(\mathscr{C}) \hookrightarrow Mor(\mathscr{D})$ is the inclusion function, then $\mathscr{C} \overset{E}{\hookrightarrow} \mathscr{D}$ is a functor, called the **inclusion functor from** $\mathscr{C}$ **to** $\mathscr{D}$.

(3) If $\tilde{\mathscr{C}}$ is a quotient category of $\mathscr{C}$, and $Q: Mor(\mathscr{C}) \to Mor(\tilde{\mathscr{C}})$ is the canonical function that assigns to each morphism $f$ its equivalence class $\tilde{f}$, then $\mathscr{C} \xrightarrow{Q} \tilde{\mathscr{C}}$ is a functor, called the **canonical** or **natural functor from** $\mathscr{C}$ **to** $\tilde{\mathscr{C}}$.

† If $X' \subset X$, $Y' \subset Y$ and $f: X \to Y$ is a function such that $f[X'] \subset Y'$, then the unique function $g: X' \to Y'$ for which the square




commutes, is called the **restriction of** $f$ **to** $(X', Y')$ and is denoted by $f|_{X'}^{Y'}$.

(4) If $\mathscr{C}$ and $\mathscr{D}$ are categories and $F\colon Mor(\mathscr{C}) \to Mor(\mathscr{D})$ is a function which takes every $\mathscr{C}$-morphism to a single identity in $Mor(\mathscr{D})$, then $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a functor, called a **constant functor from $\mathscr{C}$ to $\mathscr{D}$.**

(5) If $(\mathscr{C}, F, \mathscr{D})$ is a functor, then $(\mathscr{C}^{op}, F, \mathscr{D}^{op})$ is a functor, called the **opposite functor of** $F$ and denoted by $F^{op}$. Thus, $F$ and $F^{op}$, when considered as functions on morphism classes, are identical. (Recall that $Mor(\mathscr{C}) = Mor(\mathscr{C}^{op})$.) However, considered as *functors*, they are different since they have different domains and codomains.

(6) For any concrete category $\mathscr{C}$, there is a functor $U\colon \mathscr{C} \to$ **Set** that assigns to any object $A$, the underlying set $U(A)$ and to any morphism, the corresponding function on the underlying sets. $U$ is called the **forgetful functor** (or **underlying functor**) **on** $\mathscr{C}$. One can also easily construct the forgetful functors: **Rng** $\to$ **Ab** (which "forgets" multiplication); **Rng** $\to$ **Mon** (which "forgets" addition); $\mathbf{BanSp_2} \to \mathbf{BanSp_1}$ (which "forgets" the norm—but not the uniform topology); $\mathbf{BanSp_1} \to$ **NLinSp** (which "forgets" completeness); etc.

(7) The function $F$: **Set** $\to$ **Grp** that assigns to any set $A$ the free group with set of generators $A$, and to each set function $f$ the induced homomorphism that coincides with $f$ on the generators, is a functor, called the **free group functor.** One can similarly define the **free semigroup functor,** the **free abelian group functor,** the **free left $R$-module functor**, the **free ring functor**, etc.

(8) For each group $A$, let $A'$ be the commutator subgroup of $A$; i.e., the subgroup generated by all elements of $A$ of the form $ghg^{-1}h^{-1}$.
Define $F$: **Grp** $\to$ **Grp** by:

$$F(A) = A'$$

$$F(A \xrightarrow{f} B) = A' \xrightarrow{f|_{A'}^{B'}} B'.$$

Then $F$ is a functor, called the **commutator subgroup functor.**
Define $H$: **Grp** $\to$ **Ab** by:

$$H(A) = A/A'$$

$H(A \xrightarrow{f} B)$ is the homomorphism from $A/A'$ to $B/B'$ induced by $f$; i.e., the unique morphism for which the square

$$\begin{array}{ccc} A & \xrightarrow{\natural} & A/A' \\ {\scriptstyle f}\downarrow & & \vdots\, {\scriptstyle H(f)} \\ B & \xrightarrow{\natural} & B/B' \end{array}$$

commutes (where the horizontal arrows are natural projections). Then $H$ is a functor, called the **abelianization functor**.

(9) There is a functor $\beta$ from the category $\mathbf{CRegT_2}$ of completely regular

Hausdorff spaces to the category $\mathbf{CompT_2}$ of compact Hausdorff spaces that assigns to each space $X$ its Stone-Čech compactification $\beta X$ and to each continuous function $X \xrightarrow{f} Y$ the unique continuous function $\beta(f)$ which makes the square

$$\begin{array}{ccc} X & \hookrightarrow & \beta X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta(f)} \\ Y & \hookrightarrow & \beta Y \end{array}$$

commute. $\beta$ is called the **Stone-Čech** functor.

(10) For each pointed topological space $X$, let $\Pi_1(X)$ be the fundamental group of $X$ and for each morphism $X \xrightarrow{f} Y$ in **pTop**, let $\Pi_1(f)$ be the function from $\Pi_1(X)$ to $\Pi_1(Y)$ that assigns to each equivalence class of closed paths $[p]$ the equivalence class $[f \circ p]$. For each such $f$, $\Pi_1(f)$ is a group homomorphism. $\Pi_1$ is a functor from **pTop** to **Grp**, called the **fundamental group functor**.

(11) There is a functor $h_0$: **Top** $\to$ **Ab** that assigns to each space $X$ the free abelian group generated by the set of components of $X$. If $X \xrightarrow{f} Y$, then $h_0(f)\colon h_0(X) \to h_0(Y)$ is determined by:

$$h_0(f)(C) = \text{the component of } Y \text{ that contains } f[C]$$

(where $C$ is a component of $X$). $h_0$ is called the **0th homology functor**.

(12) If $\mathscr{C}$ is the category of chain complexes of abelian groups (3.5(5)), then for each integer $n$ there is a **homology-functor** $H_n\colon \mathscr{C} \to$ **Ab** defined by:

$$H_n((G_i, d_i)_{i\in Z}) = Ker(d_n)/Im(d_{n+1})$$

$H_n((f_i)_{i\in Z})$ = the induced homomorphism:

$$\hat{f}_n\colon Ker(d_n)/Im(d_{n+1}) \to Ker(d'_n)/Im(d'_{n+1})$$

Actually different "homology theories" of algebraic topology can essentially be obtained by defining appropriate functors from the category of topological spaces to the category of chain complexes and looking at the "compositions" of each such functor with the homology functor $H_n$.

**9.3 PROPOSITION**

If $\mathscr{A} \xrightarrow{F} \mathscr{B}$ and $\mathscr{B} \xrightarrow{G} \mathscr{C}$ are functors, then $\mathscr{A} \xrightarrow{G\cdot F} \mathscr{C}$ is a functor. □

**9.4 DEFINITION**

The functor $G \circ F$ of the above proposition is called the **composition of** $F$ **and** $G$.

The reader may have noticed that in Proposition 9.3 we regard $F$ (inaccurately) both as a function from $Mor(\mathscr{A})$ to $Mor(\mathscr{B})$ and as a functor from $\mathscr{A}$ to $\mathscr{B}$. A more precise statement of the proposition would be the following:

If $F = (\mathscr{A}, \bar{F}, \mathscr{B})$ and $G = (\mathscr{B}, \bar{G}, \mathscr{C})$ are functors, then $(\mathscr{A}, \bar{G} \circ \bar{F}, \mathscr{C})$ is a functor (which is denoted by $G \circ F$).

Because it is more descriptive and instructive, we have used the less precise "arrow-notation" for functors in Proposition 9.3. We will continue to use it when no confusion seems likely.

**9.5 DEFINITION**

A triple $(\mathscr{C}, F, \mathscr{D})$ is called a **contravariant functor from $\mathscr{C}$ to $\mathscr{D}$** if and only if $(\mathscr{C}^{op}, F, \mathscr{D})$ is a functor (or, equivalently, if and only if $(\mathscr{C}, F, \mathscr{D}^{op})$ is a functor).

Notice that a contravariant functor from $\mathscr{C}$ to $\mathscr{D}$ is usually *not* a functor from $\mathscr{C}$ to $\mathscr{D}$. Nevertheless, we will occasionally use the notation $F: \mathscr{C} \to \mathscr{D}$ for a contravariant functor from $\mathscr{C}$ to $\mathscr{D}$. For this reason, functors are sometimes called "covariant functors" when one wishes to distinguish them from contravariant functors. Observe that the notion of a contravariant functor is for the most part superfluous. The principal reason for its use is the fact that in many instances we have a built-in preference for some categories over their opposites. For example, we know just as much about $\mathbf{Set}^{op}$ and $\mathbf{Top}^{op}$ as we do about Set and **Top**. But we are psychologically more comfortable when working with the latter categories.

**9.6 EXAMPLES OF CONTRAVARIANT FUNCTORS**

(1) For any functor $F = (\mathscr{C}, \bar{F}, \mathscr{D})$ there are two associated contravariant functors,

$$F^* = (\mathscr{C}^{op}, \bar{F}, \mathscr{D}) \qquad \text{and} \qquad {}^*F = (\mathscr{C}, \bar{F}, \mathscr{D}^{op}).$$

(2) The functor $\mathscr{P}: \mathbf{Set}^{op} \to \mathbf{Set}$, which assigns to each set $A$ the set $\mathscr{P}(A)$ of all subsets of $A$ and to each function $A \xrightarrow{f} B$ the function $\mathscr{P}(B) \xrightarrow{\mathscr{P}(f)} \mathscr{P}(A)$ defined by $\mathscr{P}(f)(C) = f^{-1}[C]$, is a contravariant functor from Set to Set, called the **contravariant power-set functor.**

(3) The functor $F: \mathbf{Top}^{op} \to \mathbf{BooAlg}$ that assigns to each topological space the boolean algebra of its clopen subsets and to each continuous function $f: X \to Y$, the boolean homomorphism $F(f): F(Y) \to F(X)$ defined by $F(f)(A) = f^{-1}[A]$, is a contravariant functor from **Top** to **BooAlg**.

(4) If $\mathscr{F}$ is the category of finite dimensional vector spaces over the field $F$, then the functor $(\hat{\ })$: $\mathscr{F}^{op} \to \mathscr{F}$ that assigns to each vector space $V$, the space $\hat{V}$ of linear functionals over $F$ and to each linear transformation $f: V \to W$, the linear transformation $\hat{f}: \hat{W} \to \hat{V}$ defined by $\hat{f}(g) = g \circ f$, is a contravariant functor from $\mathscr{F}$ to $\mathscr{F}$, called the **duality functor.**

(5) The functor $C: \mathbf{Top}^{op} \to \mathbf{VecLat}$ that assigns to each topological space $X$ the vector lattice $C(X, \mathbf{R})$ of all continuous real-valued functions is a contravariant functor from **Top** to **VecLat**.

(6) The functor $C^*: \mathbf{Top}^{op} \to \mathbf{C^*\text{-}Alg}$ that assigns to each topological space $X$ the $C^*$-algebra $C^*(X, \mathbf{C})$ of all complex valued bounded continuous functions is a contravariant functor from **Top** to **C*-Alg**.

(7) Each of the cohomology functors of algebraic topology is a contravariant functor from **Top** to **Ab**; i.e., a functor from $\mathbf{Top}^{op}$ to **Ab**.

**9.7 DEFINITION**

If the domain of a functor is the product of two categories, then the functor is sometimes called a **bifunctor**. Similarly, one may define **trifunctors** and **functors of $n$-variables**.

**9.8 EXAMPLES OF BIFUNCTORS**

(1) The **cartesian product functor** $(\_\_ \times \_\_): \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$, defined by:

$$(\_\_ \times \_\_)(A, B) = A \times B$$

$$(\_\_ \times \_\_)(f, g) = f \times g: A \times B \to C \times D$$

where $(f \times g)(a, b) = (f(a), g(b))$.

(2) The **tensor product functor** $(\_\_ \otimes \_\_): \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$, defined by:

$$(\_\_ \otimes \_\_)(A, B) = A \otimes B$$

$$(\_\_ \otimes \_\_)(f, g) = f \otimes g: A \otimes B \to C \otimes D$$

where

$$(f \otimes g)(\sum_i a_i \otimes b_i) = \sum_i (f(a_i) \otimes g(b_i)).$$

(3) The **disjoint union functor** $(\_\_ \uplus \_\_): \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ defined by:

$$(\_\_ \uplus \_\_)(A, B) = A \uplus B$$

$$(\_\_ \uplus \_\_)(f, g) = f \uplus g: A \uplus B \to C \uplus D$$

where

$$(f \uplus g)(x, i) = \begin{cases} f(x) & \text{if } i = 1 \\ g(x) & \text{if } i = 2. \end{cases}$$

**9.9 THEOREM**

*If $F: \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ is a bifunctor, then*

(1) *For each $A \in Ob(\mathscr{A})$, there is an associated functor (denoted by)*

$$F(A, \_\_): \mathscr{B} \to \mathscr{C},$$

*defined by:*

$$F(A, \_\_)(B) = F(A, B)$$

*and*

$$F(A, \_\_)(h) = F(1_A, h)$$

*and*

(2) *for each $B \in Ob(\mathscr{B})$, there is an associated functor (denoted by)*

$$F(\_\_, B): \mathscr{A} \to \mathscr{C},$$

*defined by:*

$$F(\_\_, B)(A) = F(A, B)$$

*and*

$$F(\_\_, B)(g) = F(g, 1_B).$$

*Proof:* We will prove (1). (2) follows analogously. Clearly, $F(A, \_\_)(1_B) = F(1_A, 1_B)$ is an identity in $\mathscr{C}$ since $F$ is a functor. Now, if $B \xrightarrow{h} B' \xrightarrow{g} B''$ are morphisms in $\mathscr{B}$, then

$$F(A, \_\_)(g \circ h) = F(1_A \circ 1_A, g \circ h) = F((1_A, g) \circ (1_A, h))$$
$$= F(1_A, g) \circ F(1_A, h) = F(A, \_\_)(g) \circ F(A, \_\_)(h).$$

Thus, $F(A, \_\_)$ preserves identities and compositions. □

**9.10 DEFINITION**

The functor $F(A, \_\_)$ of Theorem 9.9 is called the **right associated functor with respect to** $F$ **and** $A$, and the functor $F(\_\_, B)$ is called the **left associated functor with respect to** $F$ **and** $B$.

The most important bifunctor and its associated functors will be studied separately in the next section.

## EXERCISES

9A. Verify that the following are functors:

(a) The (covariant) power set functor $\mathscr{P}$: Set → Set, defined by:

$$\mathscr{P}(A) = \text{the collection of all subsets of } A$$

$$\mathscr{P}(A \xrightarrow{f} B) = \mathscr{P}(f): \mathscr{P}(A) \to \mathscr{P}(B)$$

where $\mathscr{P}(f)(C) = f[C]$.

(b) The squaring functor $(\ )^2$: Set → Set, defined by:

$$(\ )^2(A) = A^2$$

$$(\ )^2(A \xrightarrow{f} B) = f^2: A^2 \to B^2$$

where $f^2(a_1, a_2) = (f(a_1), f(a_2))$.

(c) The $i$th projection functor for a product category

$$\pi_i: \mathscr{C}_1 \times \mathscr{C}_2 \times \cdots \times \mathscr{C}_n \to \mathscr{C}_i,$$

defined by:

$$\pi_i(f_1, f_2, \ldots, f_n) = f_i.$$

(d) The $i$th injection functor for a sum category

$$\mu_i: \mathscr{C}_i \to \mathscr{C}_1 \amalg \mathscr{C}_2 \amalg \cdots \amalg \mathscr{C}_n,$$

defined by:

$$\mu_i(f) = (f, i).$$

9B. Show that if $F: \mathscr{C} \to \mathscr{D}$ is a functor, then the image of $\mathscr{C}$ under $F$ is not necessarily a subcategory of $\mathscr{D}$. [Consider a functor from

].

9C. Show that a function between the morphism classes of two categories, which preserves compositions, is not necessarily a functor; i.e., "identity preservation" is essential. [Cf. Exercise 4B.]

9D. Show that functors which coincide on objects (identities) are not necessarily the same.

9E. Prove that if each of $\mathscr{C}$ and $\mathscr{D}$ is a category with just one object (i.e., a monoid (3.5(7)), then $F: \mathscr{C} \to \mathscr{D}$ is a functor if and only if it is a monoid homomorphism.

9F. In a manner similar to that given in Example 9.2(8), define a **torsion subgroup functor** $T$: **Ab** $\to$ **Ab** which assigns to each abelian group $A$ its torsion subgroup $\hat{A}$, and define a **torsion-free functor** $F$: **Ab** $\to$ **Ab** which assigns to each abelian group $A$ the group $A/\hat{A}$.

9G. Define a functor from the category **QOS** of quasi-ordered sets to the category **POS**, which assigns to any quasi-ordered set $X$, the partially-ordered set that is obtained by identifying those members $a$ and $b$ of $X$ for which $a \leq b$ and $b \leq a$.

9H. Let $F$: $Ob$(**NLinSp**) $\to$ $Ob$(**CompT$_2$**) be the function that assigns to each normed linear space $X$ the closed unit sphere in the space of all linear functionals on $X$ (furnished with the weak*-topology). Is $F$ the object function of some contravariant functor from **NLinSp** to **CompT$_2$**?

9I. Define a (bi)functor $(\_ \times \_)$: **Top** $\times$ **Top** $\to$ **Top** whose value at any pair of spaces is their topological product.

9J. Show that if $(G, +)$ is an abelian group (considered as a category with one object), then the direct product $G \times G$ is a product category and $\odot: G \times G \to G$ defined by $\odot(f, g) = f + g$ is a bifunctor.

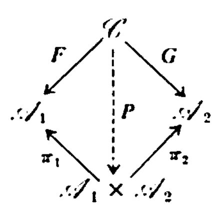
9K. Prove that if $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{C} \to \mathscr{D}$, then there exists a functor

$$F \times G: \mathscr{A} \times \mathscr{C} \to \mathscr{B} \times \mathscr{D},$$

defined by:

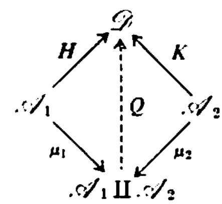
$$(F \times G)(h, k) = (F(h), G(k)).$$

9L. Prove that the product of two categories $\mathscr{A}_1 \times \mathscr{A}_2$ together with the projection functors $\pi_1: \mathscr{A}_1 \times \mathscr{A}_2 \to \mathscr{A}_1$ and $\pi_2: \mathscr{A}_1 \times \mathscr{A}_2 \to \mathscr{A}_2$ has the property that if $\mathscr{C}$ is any category and $F: \mathscr{C} \to \mathscr{A}_1$ and $G: \mathscr{C} \to \mathscr{A}_2$ are any functors, then there exists a unique functor $P: \mathscr{C} \to \mathscr{A}_1 \times \mathscr{A}_2$ such that the diagram




commutes.

9M. Prove that the sum category $\mathscr{A}_1 \amalg \mathscr{A}_2$ together with the injection functors $\mu_i: \mathscr{A}_i \to \mathscr{A}_1 \amalg \mathscr{A}_2$, $i = 1, 2$ has the property that if $\mathscr{D}$ is any category and $H: \mathscr{A}_1 \to \mathscr{D}$ and $K: \mathscr{A}_2 \to \mathscr{D}$, then there exists a unique functor $Q: \mathscr{A}_1 \amalg \mathscr{A}_2 \to \mathscr{D}$ such that the diagram




commutes.

## §10 HOM-FUNCTORS

One naturally occurring (bi)functor is so important that we wish to consider it separately.

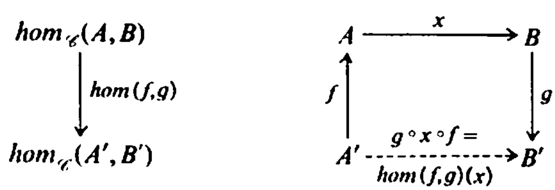
**10.1 PROPOSITION**

*If $\mathscr{C}$ is any category, then there is a functor (denoted by)*

$$hom_{\mathscr{C}}: \mathscr{C}^{op} \times \mathscr{C} \to \mathbf{Set},$$

*where $hom_{\mathscr{C}}(A, B)$ is the set of $\mathscr{C}$-morphisms from $A$ to $B$, and*

$$hom_{\mathscr{C}}(f, g)(x) = g \circ x \circ f.$$




*Proof:* Clearly

$$hom_{\mathscr{C}}(1_A, 1_B)(x) = 1_B \circ x \circ 1_A = x;$$

thus, $hom_{\mathscr{C}}(1_A, 1_B)$ is the identity function on the set $hom_{\mathscr{C}}(A, B)$. Also

$$\begin{aligned} hom_{\mathscr{C}}((f, g) \circ (h, k))(x) &= hom_{\mathscr{C}}(h \circ f, g \circ k)(x) = g \circ k \circ x \circ h \circ f \\ &= g \circ hom_{\mathscr{C}}(h, k)(x) \circ f = hom_{\mathscr{C}}(f, g)(hom_{\mathscr{C}}(h, k)(x)) \\ &= hom_{\mathscr{C}}(f, g) \circ hom_{\mathscr{C}}(h, k)(x). \end{aligned}$$

Hence, $hom_{\mathscr{C}}$ preserves identities and composition. □

**10.2 DEFINITION**

The functor $hom_{\mathscr{C}}: \mathscr{C}^{op} \times \mathscr{C} \to \mathbf{Set}$ is called the **set-valued hom-functor** (or **morphism functor**) for the category $\mathscr{C}$. The right associated functor with respect to $hom_{\mathscr{C}}$ and $A$, i.e., $hom_{\mathscr{C}}(A, \_\_): \mathscr{C} \to \mathbf{Set}$, is called the **covariant hom-functor of $\mathscr{C}$ with respect to $A$**; and the left associated functor, i.e.,

$$hom_{\mathscr{C}}(\_\_, A): \mathscr{C}^{op} \to \mathbf{Set},$$

is called the **contravariant hom-functor of $\mathscr{C}$ with respect to $A$**.

Occasionally, for simplicity, the subscript $\mathscr{C}$ naming the category is suppressed when denoting the *hom*-functor. Notice that for a morphism $f$

$$hom(A,\_)(f) = hom(A,f) = f\circ\_;\text{ i.e., } hom(A,f)(x) = f\circ x,$$

whereas

$$hom(\_,A)(f) = hom(f,A) = \_\circ f;\text{ i.e., } hom(f,A)(x) = x\circ f.$$

**10.3 PROPOSITION**

*For any category $\mathscr{C}$ and any $\mathscr{C}$-object $A$, we have*

$$hom_{\mathscr{C}}(\_,A) = hom_{\mathscr{C}^{op}}(A,\_).$$

*In other words, the contravariant hom-functor $hom(\_,A)\colon \mathscr{C}^{op}\to$ **Set** of $\mathscr{C}$ with respect to $A$ is identical with the covariant hom-functor $hom(A,\_)\colon \mathscr{C}^{op}\to$ **Set** of $\mathscr{C}^{op}$ with respect to $A$.* □

Since the contravariant *hom*-functors of $\mathscr{C}$ are exactly the covariant *hom*-functors of $\mathscr{C}^{op}$, in the sequel we will formally need to investigate only covariant *hom*-functors.

### Internal Hom-Functors

In some instances, the set $hom_{\mathscr{C}}(A, B)$ can be supplied in a natural way with a structure so that it itself can be considered as an object of a category other than **Set**. For example, if $A$ and $B$ are objects in **Ab**, then $hom(A, B)$ can be regarded as an abelian group (if we define $(f+g)(a)$ to be $f(a)+g(a)$). Since, moreover, for all morphisms $f$ and $g$ of **Ab**, the function $hom(f, g)$ turns out to be an abelian group homomorphism, the *hom*-functor can in this case be considered as a functor from $\mathbf{Ab}^{op}\times\mathbf{Ab}$ to **Ab**. In this instance and in similar cases, we will use the notation "*Hom*" rather than "*hom*". Note that if $Hom_{\mathscr{C}}\colon \mathscr{C}^{op}\times\mathscr{C}\to\mathscr{D}$ and if $U\colon\mathscr{D}\to\mathbf{Set}$ is the forgetful functor, then $hom = U\circ Hom$.

**10.4 EXAMPLES**

(1) $Hom\colon \mathbf{Ab}^{op}\times\mathbf{Ab}\to\mathbf{Ab}$.

(2) $Hom\colon R\text{-}\mathbf{Mod}^{op}\times R\text{-}\mathbf{Mod}\to R\text{-}\mathbf{Mod}$ (where $R$ is commutative).

(3) $Hom\colon \mathbf{Top}^{op}\times\mathbf{Top}\to\mathbf{Top}$ (where $Hom(A, B)$ has the compact-open topology).

(4) $Hom\colon \mathbf{NLinSp}^{op}\times\mathbf{NLinSp}\to\mathbf{NLinSp}$ (where $Hom(A, B)$ has the norm defined by

$$\|f\| = sup\{\|f(x)\| \mid \|x\| = 1\}).$$

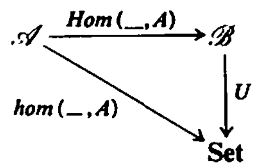
**10.5 DEFINITION**

If $\mathscr{B}$ is a concrete category with forgetful functor $U\colon\mathscr{B}\to\mathbf{Set}$, then a contravariant functor $\mathscr{A}\xrightarrow{F}\mathscr{B}$ is called a **hom-type functor** provided that there exists some $A\in Ob(\mathscr{A})$ such that $hom(\_,A) = U\circ F$. In this case, $F$ will usually be denoted by $Hom(\_,A)$.

Often *hom*-type functors arise when there is an object $A$ that can be considered simultaneously as an object of each of the categories $\mathscr{A}$ and $\mathscr{B}$. Many duality theories hinge upon these *hom*-type functors.

**10.6 EXAMPLES OF HOM-TYPE FUNCTORS**

I.e., contravariant functors of the form $Hom(\_\_, A): \mathscr{A} \to \mathscr{B}$, where the triangle




commutes.

| | $\mathscr{A}$ | $\mathscr{B}$ | $A$ | $Hom(X, A)$ is called: |
|---|---|---|---|---|
| (1) | $R$-**Mod** | **Mod**-$R$ | $R$ | The dual module of $X$ or the module of all linear functionals on $X$ |
| (2) | **LinTop** | **C-Mod** | **C** | The adjoint (or dual, or conjugate) module of the linear topological space $X$ (or the module of all linear functionals on $X$) |
| (3) | **NLinSp** | $\mathbf{BanSp_1}$ | **C** | The conjugate (or dual) Banach space of the normed linear space $X$ |
| (4) | **LCAb**† | **LCAb** | **R/Z** | The character group of the locally compact abelian group $X$ |
| (5) | **Ab** | **CompAb**†† | **R/Z** | The character group of the abelian group $X$ |
| (6) | **CompAb** | **Ab** | **R/Z** | The character group of the compact abelian group $X$ |
| (7) | **Top** | **BooAlg** | 2†††† | The dual algebra of $X$ or the boolean algebra of clopen subsets of $X$ |
| (8) | **BooAlg** | **BooSp**††† | 2†††† | The Stone space or dual space of the boolean algebra $X$ |
| (9) | **Top** | **Rng** | **R** | The ring of real-valued continuous functions on $X$ |
| (10) | **CBanAlg** | $\mathbf{CompT_2}$ | **C** | The carrier space (or maximal ideal space) of the commutative Banach algebra $X$ |

† **LCAb** is the category of locally compact abelian groups and continuous homomorphisms.
†† **CompAb** is the category of compact abelian groups and continuous homomorphisms.
††† **BooSp** is the category of boolean spaces (i.e., totally disconnected compact Hausdorff spaces) and continuous functions.
†††† 2 is considered as either the two-element discrete space or the two-element boolean algebra.

## EXERCISES

10A. Show that there exist two natural (tri)functors

$$F, G\colon \mathbf{Set}^{op} \times \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$$

where

$$F(A, B, C) = hom(A \times B, C),$$

and

$$G(A, B, C) = hom(A, hom(B, C)).$$

That is, describe how these functors act on morphisms.

10B. Every functor $F\colon \mathscr{C} \to \mathscr{D}$ has two associated set-valued bifunctors:

$$hom(F\_\_, \_\_)\colon \mathscr{C}^{op} \times \mathscr{D} \to \mathbf{Set}$$

and

$$hom(\_\_, F\_\_)\colon \mathscr{D}^{op} \times \mathscr{C} \to \mathbf{Set},$$

where

$$hom(F\_\_, \_\_)(C, D) = hom(F(C), D)$$

and

$$hom(\_\_, F\_\_)(D, C) = hom(D, F(C)).$$

Describe how these functors act on morphisms.

## §11 CATEGORIES OF CATEGORIES

We have already seen that functors assume the role of "morphisms between categories", i.e., the composition of functors is a functor (9.3), and, since their composition is the usual composition of functions, composition is associative and identity functors behave like identities with respect to the composition. Because of this, one is tempted to form the "category of all categories". However, two technical difficulties arise. First, the "category of all categories" would have objects such as Set, Grp, and Top, which are *not* sets, so that the conglomerate of all objects in the category would not be a class (1.2(1)). This violates part (i) of the definition of a category (3.1). Secondly, given any two categories $\mathscr{C}$ and $\mathscr{D}$, it is not generally true that the conglomerate of all functors from $\mathscr{C}$ to $\mathscr{D}$ forms a set. This violates part (4) of the definition of a category. However, if we restrict our attention to categories that are *sets*, i.e., to small categories, then the above problems are eliminated.

**11.1 PROPOSITION**

*There exists a category* $(\mathcal{O}, \mathcal{M}, dom, cod, \circ)$ *where* $\mathcal{O}$ *is the class of all small categories,* $\mathcal{M}$ *is the class of all functors between small categories, dom and cod are functions that assign to each* $F \in \mathcal{M}$ *its domain and codomain, respectively, and* $\circ$ *is the usual composition of functors in* $\mathcal{M}$.

*Proof:* Every functor between small categories is a set (Exercise 11A). Thus, the required classes and functions can be formed. Since the composition of functors is really function composition, the "matching", "associativity", and "identity existence" conditions are easily verified (3.1). Thus, we need only show the "smallness of the morphism class" condition; i.e., if

$$\mathscr{C} = (\mathcal{O}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}}, dom_{\mathscr{C}}, cod_{\mathscr{C}}, \circ_{\mathscr{C}})$$

and

$$\mathscr{D} = (\mathcal{O}_{\mathscr{D}}, \mathscr{M}_{\mathscr{D}}, dom_{\mathscr{D}}, cod_{\mathscr{D}}, \circ_{\mathscr{D}})$$

are small categories, then

$$\{F \mid F \text{ is a functor from } \mathscr{C} \text{ to } \mathscr{D}\}$$

is a set. Now, since $\mathscr{C}$ and $\mathscr{D}$ are sets, $\mathscr{M}_{\mathscr{C}}$ and $\mathscr{M}_{\mathscr{D}}$ must be sets, so that $\mathscr{F} = \{g \mid g \text{ is a function from } \mathscr{M}_{\mathscr{C}} \text{ to } \mathscr{M}_{\mathscr{D}}\}$ is a set (1.1(3)). Hence, $\{\mathscr{C}\} \times \mathscr{F} \times \{\mathscr{D}\}$ is a set (1.1(3)). By the definition of functor (9.1), each functor from $\mathscr{C}$ to $\mathscr{D}$ belongs to $\{\mathscr{C}\} \times \mathscr{F} \times \{\mathscr{D}\}$. Thus, by 1.1(1),

$$\{F \mid F \text{ is a functor from } \mathscr{C} \text{ to } \mathscr{D}\}$$

is a set. □

**11.2 DEFINITION**

The category given by the previous proposition is called the **category of small categories** and is denoted by **Cat**.

**Cat** is actually quite large; for example, each of the categories **Set**, **POS**, **Mon**, **Grp**, and **Ab** can be fully embedded in it (see Exercise 12F). It is nevertheless unfortunate that we cannot form the "category of all categories". Also, as we shall see later, other constructions lead to entities that would be categories were it not for the two "smallness" conditions required for categories, namely:

(1) $Ob(\mathscr{C})$ and $Mor(\mathscr{C})$ must be classes, and

(2) For each pair $(A, B)$ of $\mathscr{C}$-objects, $hom_{\mathscr{C}}(A, B)$ must be a set.

For this reason, we consider the following more general notion of a quasicategory.

**11.3 DEFINITION**

A **quasicategory** is a quintuple $\mathscr{C} = (\mathcal{O}, \mathscr{M}, dom, cod, \circ)$ where

(i) $\mathcal{O}$ and $\mathscr{M}$ are conglomerates,

(ii) *dom* and *cod* are functions from $\mathscr{M}$ to $\mathcal{O}$; and

(iii) $\circ$ is a function from

$$D = \{(f, g) \in \mathscr{M} \times \mathscr{M} \mid dom(f) = cod(g)\}$$

into $\mathscr{M}$;

such that:

(1) If $(f \circ g)$ is defined (i.e., if $(f, g) \in D$), then $dom(f \circ g) = dom(g)$ and $cod(f \circ g) = cod(f)$;

(2) If $f \circ g$ and $h \circ f$ are defined, then $h \circ (f \circ g) = (h \circ f) \circ g$;

(3) For each $A \in \mathcal{O}$, there exists some $e \in \mathcal{M}$ such that $dom(e) = A = cod(e)$ and
 (a) $f \circ e = f$ whenever $f \circ e$ is defined, and
 (b) $e \circ g = g$ whenever $e \circ g$ is defined.
(Compare this with Definition 3.1.)

**11.4 PROPOSITION**
*Every category is a quasicategory.* □

**11.5 PROPOSITION**
*There exists a quasicategory* $(\mathfrak{C}, \mathfrak{F}, dom, cod, \circ)$ *where* $\mathfrak{C}$ *is the conglomerate of all categories,* $\mathfrak{F}$ *is the conglomerate of all functors between categories, dom and cod are functions that assign to each functor its domain and codomain, respectively; and* $\circ$ *is the composition of functors.* □

**11.6 DEFINITION**
The quasicategory described in the above proposition is called the **quasicategory of all categories** and is denoted by $\mathscr{CAT}$.

One of the primary uses of the notion of a quasicategory (as opposed to that of a category) is that it allows one to talk about "naturally" occurring entities such as $\mathscr{CAT}$. It should be noted that most of the notions associated with categories, e.g., monomorphisms, terminal objects, functors, etc., can be defined for quasicategories as well. (An exception, of course, is the fact that quasicategories in general lack *hom*-functors into **Set**.) Actually, our main use for quasicategories will be as a device that allows more convenient expression. Thus, since quasicategories are not the main object of our study, we will not often be concerned with their "internal workings". Also, we will never have a need to consider something like the "quasicategory of all quasicategories". This is fortunate since, if we did, our foundations would have to be revised to handle it—we would find ourselves in another Russell-like paradox (see Exercise 11B). To keep our attention close to the motivating examples, i.e., **Set**, **Grp**, and **Top**, in that which follows, we will only consider categories (in the restricted sense) except in those instances where using quasicategories materially simplifies or clarifies matters.

## EXERCISES

11A. Prove that every functor between small categories is a set.

11B. Show that one may not form the "quasicategory of all quasicategories" by obtaining a Russell-like paradox from the assumption that one can. [Consider the full sub-quasicategory of all quasicategories that are not objects of themselves.]

11C. *Special Morphisms in* $\mathscr{CAT}$
(a) Show that the monomorphisms in $\mathscr{CAT}$ are precisely those functors whose underlying functions between morphism classes are injective.
(b) Show that each functor whose underlying function on morphism classes is surjective is an epimorphism in $\mathscr{CAT}$.

(c) Let **2** be the category $\cdot \xrightarrow{f} \cdot$, let **N** be the monoid of natural numbers considered as a category, and let $F: \mathbf{2} \to \mathbf{N}$ be the functor such that $F(f) = 1$. Show that $F$ is an epimorphism in $\mathscr{CAT}$ even though it is not surjective on the underlying morphism classes.
(d) Determine the constant morphisms in $\mathscr{CAT}$.

11D. Prove that **Cat** is well-powered, but that $\mathscr{CAT}$ is not.

## §12 PROPERTIES OF FUNCTORS

In this section, we will consider certain properties enjoyed by all functors as well as other properties possessed by special functors.

**12.1 DEFINITION**

(1) A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is said to **preserve the categorical property** $P$ provided that the image under $F$ of each morphism (or object, or diagram) in $\mathscr{C}$ with property $P$ has property $P$ in $\mathscr{D}$.

(2) $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is said to **reflect the categorical property** $P$ provided that whenever the image under $F$ of a morphism (or object, or diagram) in $\mathscr{C}$ has property $P$ in $\mathscr{D}$, then the morphism (or object, or diagram) must have property $P$ in $\mathscr{C}$.

**12.2 PROPOSITION**

*Every functor preserves identities, isomorphisms, sections, retractions and commutative triangles.*

*Proof:* That identities and commutative triangles are preserved follows from the definition of functor (9.1). Suppose that $\mathscr{C} \xrightarrow{F} \mathscr{D}$ and $f, g \in Mor(\mathscr{C})$ such that $f \circ g = 1_A$. Then,

$$1_{F(A)} = F(1_A) = F(f \circ g) = F(f) \circ F(g).$$

Thus, $F(g)$ must be a $\mathscr{B}$-section and $F(f)$ must be a $\mathscr{B}$-retraction. Hence, $F$ preserves sections and retractions. Consequently, $F$ must preserve isomorphisms. □

The fact that functors preserve isomorphisms implies that if $X$ and $Y$ are topological spaces such that for some $n$, the homology groups $H_n(X)$ and $H_n(Y)$ are not isomorphic, then $X$ and $Y$ are not homeomorphic. This fact provides a relatively simple way of showing, for example, that the torus, $S^1 \times S^1$, is not homeomorphic with the 2-sphere, $S^2$. Being able to solve problems by such techniques was one of the major reasons for the emergence of algebraic topology.

**12.3 PROPOSITION**

*If $\mathscr{C}$ and $\mathscr{D}$ are connected categories, $\mathscr{C}$ has a terminal object $X$, and $F: \mathscr{C} \to \mathscr{D}$ is a functor, then the following are equivalent:*

(1) *$F$ preserves constant morphisms.*

(2) *$F$ preserves the terminal object.*

*Proof:*

*(1)* ⇒ *(2)*. Suppose that $D \overset{h}{\underset{g}{\rightrightarrows}} F(X)$. Since $X \xrightarrow{1_X} X$ is a constant morphism (8D), its image $F(1_X) = 1_{F(X)}$ must be constant. Hence, $1_{F(X)} \circ h = 1_{F(X)} \circ g$; so that $h = g$. Thus, $F(X)$ is a terminal object.

*(2)* ⇒ *(1)*. Suppose that $C \xrightarrow{f} C'$ is a constant morphism. Then $f$ can be factored through $X$ (8.5); i.e.,

$$C \xrightarrow{f} C' = C \xrightarrow{g} X \xrightarrow{h} C'.$$

But this implies that

$$F(C) \xrightarrow{F(f)} F(C') = F(C) \xrightarrow{F(g)} F(X) \xrightarrow{F(h)} F(C').$$

Thus, since $F(X)$ is a terminal object, $F(f)$ must be constant (8.5). □

**12.4 DUALITY WITH FUNCTORS**

As was pointed out earlier, every categorical statement or property has a dual that is obtained by replacing each category by its opposite and translating this back into a statement or property involving the original categories. If such a statement or property involves a functor $F: \mathscr{C} \to \mathscr{D}$, then in the dualization process, we begin with the functor $F^{op}: \mathscr{C}^{op} \to \mathscr{D}^{op}$. (Notice in particular that the direction of the functor is not reversed.) We now illustrate this process by forming the dual of Proposition 12.3. Replacing each category by its opposite, we have:

*If $\mathscr{C}^{op}$ and $\mathscr{D}^{op}$ are connected categories, $\mathscr{C}^{op}$ has a terminal object $X$, and $F^{op}: \mathscr{C}^{op} \to \mathscr{D}^{op}$ is a functor, then the following are equivalent:*

(1) *$F^{op}$ preserves constant morphisms.*

(2) *$F^{op}$ preserves the terminal object.*

Translating this back into a statement about $\mathscr{C}$ and $\mathscr{D}$, we have the dual of Proposition 12.3:

*If $\mathscr{C}$ and $\mathscr{D}$ are connected categories, $\mathscr{C}$ has an initial object $X$, and $F: \mathscr{C} \to \mathscr{D}$ is a functor, then the following are equivalent:*

(1) *$F$ preserves coconstant morphisms.*

(2) *$F$ preserves the initial object.*

Notice that from the duality principle, it follows that if all functors have a certain property, then all have the dual property. For example, knowing that all functors preserve sections, we could have concluded using duality alone that all functors preserve retractions (cf. Proposition 12.2). It should be mentioned that whenever a categorical statement or property involves just two categories and one functor between them, other "dual" statements may be formed by replacing categories by their opposites *one at a time* and translating back into statements about the original categories. Since these statements involve contravariant

functors, they are called "contravariant duals" of the original statement. For example, the two contravariant duals of Proposition 12.3 are the following:
*If $\mathscr{C}$ and $\mathscr{D}$ are connected, X is a $\mathscr{C}$-initial object and $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a contravariant functor from $\mathscr{C}$ to $\mathscr{D}$, then the following are equivalent:*

(1) *F carries $\mathscr{C}$-coconstants to $\mathscr{D}$-constants.*
(2) *$F(X)$ is a $\mathscr{D}$-terminal object.*

*If $\mathscr{C}$ and $\mathscr{D}$ are connected, X is a $\mathscr{C}$-terminal object and $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a contravariant functor from $\mathscr{C}$ to $\mathscr{D}$, then the following are equivalent:*

(1) *F carries $\mathscr{C}$-constants to $\mathscr{D}$-coconstants.*
(2) *$F(X)$ is a $\mathscr{D}$-initial object.*

As before, the formation of duals will, for the most part, be left as a standard implicit exercise for the reader. In the sequel, when we speak of "the" dual of a statement or property involving functors, we will mean the "covariant dual".

**12.5 DEFINITION**

A functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$ is said to be:

(1) **full** provided that each *hom*-set restriction

$$F\,|^{hom(FA,FA')}_{hom(A,A')}$$

of $F$ is surjective.

(2) **faithful** provided that each *hom*-set restriction of $F$ is injective.

(3) (an) **embedding** provided that $F\colon Mor(\mathscr{A}) \to Mor(\mathscr{B})$ is an injective function.

(4) **dense**† provided that for each $B \in Ob(\mathscr{B})$ there exists some $A \in Ob(\mathscr{A})$ such that $F(A)$ is isomorphic to $B$.

One should be careful to distinguish between embeddings and faithful functors. For example, each of the forgetful functors from **Grp**, **Mon**, **Top**, or $R$-**Mod** into **Set** is faithful, but none is an embedding. It is easily seen that a functor is an embedding provided that it is both faithful and "one-to-one on objects" (i.e., on identities).

Notice that a functor $F$ is full (resp. faithful, (an) embedding, dense) if and only if $F^{op}$ has the same property. Thus, for example, if all faithful functors preserve some property, they must also preserve the dual property.

**12.6 EXAMPLES**

(1) Every canonical functor from a category to a quotient category is both full and dense.

(2) Every inclusion functor of a subcategory into a category is an embedding.

(3) A category $\mathscr{C}$ is a quasi-ordered class (in the sense of 3.5(6)) if and only if each functor with domain $\mathscr{C}$ is faithful.

† A functor with this property is sometimes called "**representative**".

(4) Every forgetful functor is faithful.

(5) The forgetful functor from **Field** to **Set** is not full and not dense. [There is no field on a six-element set.]

(6) The forgetful functors from **Grp** to **Set** and **Ab** to **Set** are dense, but not full.

(7) The inclusion functor from $\mathbf{Top}_2$ to **Top** is full, but not dense.

(8) The abelianization functor from **Grp** to **Ab** (9.2(8)) is dense, but not full or faithful.

We have seen that with any concrete category $\mathscr{C}$, there can be associated in a natural way a category $\mathscr{A}$ and a faithful functor $U: \mathscr{A} \to$ **Set**. Conversely, every pair $(\mathscr{A}, U)$ where $\mathscr{A}$ is a category and $U: \mathscr{A} \to$ **Set** is a faithful functor, gives rise to a concrete category. Thus, the two concepts are essentially the same. In the future, we will use the term "concrete category" in the following sense:

### 12.7 SECOND DEFINITION OF CONCRETE CATEGORY

A **concrete category** is a pair $(\mathscr{A}, U)$ where $\mathscr{A}$ is a category and $U: \mathscr{A} \to$ **Set** is a faithful functor.

### 12.8 PROPOSITION

*Every faithful functor* $\mathscr{A} \xrightarrow{F} \mathscr{B}$ *reflects monomorphisms, epimorphisms, bimorphisms, constant morphisms, coconstant morphisms, zero morphisms, and commutative triangles.*

*Proof:*

(1) Suppose that $F(A) \xrightarrow{F(f)} F(A')$ is a $\mathscr{B}$-monomorphism. If $A'' \underset{k}{\overset{h}{\rightrightarrows}} A$ are morphisms such that $f \circ h = f \circ k$, then $F(f) \circ F(h) = F(f) \circ F(k)$. Since $F(f)$ is left-cancellable, $F(h) = F(k)$. Thus, since

$$F\,\big|^{hom(FA'',FA)}_{hom(A'',A)}$$

is injective, $k = h$. Therefore, $F$ reflects monomorphisms.

(2) If $F(A) \xrightarrow{F(f)} F(A')$ is a $\mathscr{B}$-constant morphism and if $A'' \underset{s}{\overset{r}{\rightrightarrows}} A$, then

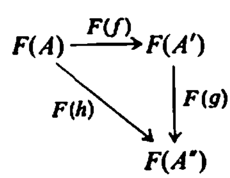
$$F(f) \circ F(r) = F(f) \circ F(s),$$

so that $F(f \circ r) = F(f \circ s)$. Since

$$F\,\big|^{hom(FA'',FA')}_{hom(A'',A')}$$

is injective, $f \circ r = f \circ s$; i.e., $f$ is constant. Thus, $F$ reflects constants.

(3) If the triangle



commutes, then $F(h) = F(g) \circ F(f) = F(g \circ f)$. Since

$$F\Big|_{hom(A, A'')}^{hom(FA, FA'')}$$

is injective, $h = g \circ f$. Thus, $F$ reflects commutative triangles.

The remaining parts of the proposition follow from the fact that faithfulness is a self-dual concept. □

**12.9 PROPOSITION**

*Every functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$ that is full and faithful reflects sections, retractions, and isomorphisms.*

*Proof:* If $F(A) \xrightarrow{F(f)} F(A')$ is a $\mathscr{B}$-section, then there is some $\mathscr{B}$-morphism $F(A') \xrightarrow{h} F(A)$ such that $h \circ F(f) = 1_{F(A)}$. Since $F$ is full, there is some $\mathscr{A}$-morphism $g: A' \to A$ such that $F(g) = h$. Now $F(g \circ f) = F(g) \circ F(f) = h \circ F(f) = 1_{F(A)} = F(1_A)$. Thus, since $F$ is faithful, $g \circ f = 1_A$ and so $f$ is an $\mathscr{A}$-section. Hence, $F$ reflects sections. That $F$ reflects retractions follows from the fact that fullness and faithfulness are both self-dual concepts. □

**12.10 THEOREM**

*Every functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$ that is full, faithful, and dense preserves and reflects monomorphisms, epimorphisms, bimorphisms, constant morphisms, coconstant morphisms, zero morphisms, sections, retractions, isomorphisms, and commutative triangles.*

*Proof:* By the above two propositions and duality, we need only show that $F$ preserves monomorphisms and constant morphisms. To do this, we start with the following:

Let $A' \xrightarrow{f} A$ be an $\mathscr{A}$-morphism and let $B \underset{h}{\overset{g}{\rightrightarrows}} F(A')$ be any $\mathscr{B}$-morphisms.

Since $F$ is dense, there is an $\mathscr{A}$-object $A''$ and a $\mathscr{B}$-isomorphism $F(A'') \xrightarrow{s} B$. Since $F$ is full, there are $\mathscr{A}$-morphisms $q$ and $r$ such that $F(q) = g \circ s$ and $F(r) = h \circ s$.

$$\begin{array}{ccc} A'' & \qquad & F(A'') \\ q \downarrow \; \downarrow r & & \downarrow s \\ A' & & B \\ \downarrow f & & g \downarrow \; \downarrow h \\ A & & F(A') \\ & & \downarrow F(f) \\ & & F(A) \end{array}$$

*Monomorphism Preservation*
If $f$ is a monomorphism, then

$$F(f) \circ g = F(f) \circ h \Rightarrow$$
$$F(f) \circ g \circ s = F(f) \circ h \circ s \Rightarrow$$
$$F(f) \circ F(q) = F(f) \circ F(r) \Rightarrow$$
$$F(f \circ q) = F(f \circ r) \Rightarrow$$
$$f \circ q = f \circ r \text{ (since } F \text{ is faithful)} \Rightarrow$$
$$q = r \text{ (since } f \text{ is a monomorphism)} \Rightarrow$$
$$F(q) = F(r) \Rightarrow$$
$$g \circ s = h \circ s \Rightarrow$$
$$g = h \text{ (since } s \text{ is an epimorphism)}$$

Thus, $F(f)$ is a monomorphism.

*Constant Morphism Preservation*
If $f$ is a constant morphism, then

$$f \circ q = f \circ r \Rightarrow$$
$$F(f) \circ F(q) = F(f) \circ F(r) \Rightarrow$$
$$F(f) \circ g \circ s = F(f) \circ h \circ s \Rightarrow$$
$$F(f) \circ g = F(f) \circ h \text{ (since } s \text{ is an epimorphism).}$$

Hence, $F(f)$ is a constant morphism. □

**12.11 PROPOSITION**

*The composition of full (resp. faithful, embedding, dense) functors is full (resp. faithful, embedding, dense).* □

**12.12 PROPOSITION**

*A subcategory of a category is a full (resp. dense) subcategory if and only if the inclusion functor is full (resp. dense).* □

**Properties of hom-Functors**

**12.13 PROPOSITION**

*Each covariant hom-functor, $hom(A, \_)$, preserves monomorphisms.*

*Proof:* If $B \xrightarrow{f} C$ is a monomorphism, then since $f$ is left-cancellable,

$$(hom(A, \_)(f))(x) = (hom(A, \_)(f))(y) \Rightarrow f \circ x = f \circ y \Rightarrow x = y.$$

Thus, $\hom(A, \_)(f)$ is an injective function; i.e., a Set-monomorphism. □
(See also Corollary 29.4.)

**12.14 DEFINITION**

A $\mathscr{C}$-object $P$ is called **$\mathscr{C}$-projective** if and only if the functor

$$hom_{\mathscr{C}}(P, \_\_): \mathscr{C} \to \mathbf{Set}$$

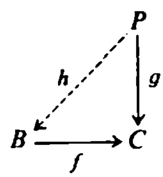
preserves epimorphisms. Dually, $Q$ is called **$\mathscr{C}$-injective** if and only if

$$hom_{\mathscr{C}}(\_\_, Q): \mathscr{C}^{op} \to \mathbf{Set}$$

preserves epimorphisms (see Proposition 10.3).

**12.15 PROPOSITION (CHARACTERIZATION OF PROJECTIVE OBJECTS)**

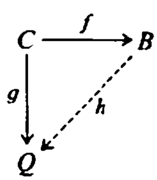
*An object $P$ is $\mathscr{C}$-projective provided that for each $\mathscr{C}$-epimorphism $B \xrightarrow{f} C$ and each morphism $P \xrightarrow{g} C$, there exists a morphism $P \xrightarrow{h} B$ such that the triangle*




*commutes.* □

**12.16** Notice that the above gives a completely "internal characterization" of projective objects in $\mathscr{C}$. Using duality, it is then easy to form an internal characterization of injective objects; namely:

*An object $Q$ is $\mathscr{C}$-injective provided that for each $\mathscr{C}$-monomorphism $C \xrightarrow{f} B$ and each morphism $C \xrightarrow{g} Q$, there exists a morphism $B \xrightarrow{h} Q$ such that the triangle*




*commutes.*

These internal characterizations are often easier to work with when determining the projectives and injectives in particular categories. Notice that some categorical concepts that have been defined "internally" also have "external" *hom*-functor characterizations and hence could have been defined externally; e.g.,

*$A \xrightarrow{f} B$ is a monomorphism if and only if $hom(C, \_\_)(f)$ is injective for each $\mathscr{C}$-object $C$.*

**12.17 EXAMPLES**

(1) A left $R$-module is categorically ($R$-**Mod**)-projective if and only if it is a projective $R$-module and is ($R$-**Mod**)-injective if and only if it is an injective $R$-module. [This provides the motivation for our terminology.]

(2) If $R$ is a principal ideal domain, then $A$ is ($R$-**Mod**)-projective if and only if it is free and is ($R$-**Mod**)-injective if and only if it is divisible.

(3) In **Grp**, **BooAlg**, and $R$-**Mod**, the projective objects are precisely the retracts of free objects, i.e., of free groups, free Boolean algebras and free modules (cf. 31.10).

(4) A boolean algebra is **BooAlg**-injective if and only if it is complete.

(5) A boolean space is **BooSp**-projective if and only if it is extremally disconnected and is **BooSp**-injective if and only if it is a retract of a Cantor space.†

(6) A topological space is **Top**-projective if and only if it is discrete and is **Top**-injective if and only if it is indiscrete and non-empty.

**12.18 DEFINITION**

A $\mathscr{C}$-object $S$ is called a **separator for** $\mathscr{C}$ if and only if the functor

$$hom(S, \_\_): \mathscr{C} \to \mathbf{Set}$$

is faithful.

DUAL NOTION: **coseparator for** $\mathscr{C}$.

**12.19 PROPOSITION (CHARACTERIZATION OF SEPARATORS)**

*A $\mathscr{C}$-object $S$ is a separator for $\mathscr{C}$ if and only if whenever $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are distinct $\mathscr{C}$-morphisms, there exists a $\mathscr{C}$-morphism $S \xrightarrow{x} A$ such that*

$$S \xrightarrow{x} A \xrightarrow{f} B \neq S \xrightarrow{x} A \xrightarrow{g} B.$$

*Proof:* If $hom(S, \_\_)$ is faithful and $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are distinct, then $hom(S, f) \neq hom(S, g)$. But these are functions with domain $hom(S, A)$. Hence, there is some $x \in hom(S, A)$ such that $hom(S, f)(x) \neq hom(S, g)(x)$; i.e., $f \circ x \neq g \circ x$. The converse can be obtained by reversing each of these implications. □

**12.20** The above characterization motivates our terminology and gives an "internal" description of separators. Dualizing it, we obtain the following: *A $\mathscr{C}$-object $C$ is a coseparator for $\mathscr{C}$ if and only if whenever $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are distinct $\mathscr{C}$-morphisms, there exists a $\mathscr{C}$-morphism $B \xrightarrow{x} C$ such that*

$$A \xrightarrow{f} B \xrightarrow{x} C \neq A \xrightarrow{g} B \xrightarrow{x} C.$$

**12.21 EXAMPLES**

(1) The separators for **Set** are precisely the non-empty sets.

(2) The separators for **Top**, $\mathbf{Top_2}$ or $\mathbf{CompT_2}$ are precisely the non-empty spaces in these categories.

(3) The group of integers **Z** under addition is a separator for **Grp** and for **Ab**.

† A space is **extremally disconnected** provided that the closure of every open set is open and is a **Cantor space** if and only if it is homeomorphic with some power of the two-point discrete space.

(4) The monoid of natural numbers N under addition is a separator for **Mon**.

(5) For every ring $R$, $R$ is a separator for $R$-**Mod**.

(6) The sets with at least two elements are precisely the coseparators for **Set**.

(7) The spaces with non-trivial indiscrete subspaces are precisely the coseparators for **Top**.

(8) The two-element space that is not discrete and not indiscrete is a coseparator for the category of $T_0$-spaces.

(9) The two-element discrete space is a coseparator for the category of zero-dimensional $T_0$-spaces.

(10) The circle group **R**/**Z** is a coseparator for **Ab**, and considered as a compact group, it is a coseparator for the category of locally compact abelian groups.

(11) The two-element boolean algebra is a coseparator for **BooAlg**.

(12) The closed unit interval is a coseparator for $\mathbf{CompT_2}$ or for $\mathbf{CRegT_2}$.

(13) The complex numbers, **C**, regarded as a Banach space is a coseparator for $\mathbf{BanSp_1}$ and for $\mathbf{BanSp_2}$ (Hahn-Banach theorem).

(14) None of the categories $\mathbf{Top_2}$, **Rng**, or **SGrp** has a coseparator.†

## EXERCISES

12A. Give an example of a functor which does not preserve monomorphisms.

12B. Prove that if $\mathscr{A}$ and $\mathscr{B}$ have zero objects and $\mathscr{A} \xrightarrow{F} \mathscr{B}$, then the following are equivalent:

(a) $F$ preserves constant morphisms.

(b) $F$ preserves coconstant morphisms.

(c) $F$ preserves zero morphisms.

(d) $F$ preserves zero objects.

12C. Give an example of a full functor which is not a surjection and a dense functor which is not a surjection on objects.

12D. Prove that when the composition $F \circ G$ of functors is full, then $F$ is full on the image of $G$; and when it is faithful, then $G$ is faithful.

12E. Let $F: \mathscr{A} \to \mathscr{B}$ be a functor that has one of the following properties:

(a) $F$ is full.

(b) $F$ is one-to-one on objects.

Prove that the "image of $\mathscr{A}$" under $F$ is a subcategory of $\mathscr{B}$ (cf. Exercise 9B).

12F. (a) Prove that the categories **Set**, **POS**, **Mon**, **Grp**, and **Ab** can be fully embedded in **Cat** (i.e., for each of these categories, $\mathscr{A}$, there is a full embedding functor $F: \mathscr{A} \to$ **Cat**). [Consider, e.g., the subcategory of all small discrete categories and the subcategory of all small one-object categories.]

† To prove this, use the following facts:
(1) for each $T_2$-space $X$ there is a $T_2$-space $Y$ with more than one point such that every continuous map from $Y$ to $X$ is constant.
(2) There exist arbitrarily large fields and arbitrarily large simple groups.

(b) Prove that a category is pointed if and only if it can be fully embedded in a category with a zero object.

12G. Prove that $f: A \to B$ is an epimorphism in Mon if and only if $f$ (considered as a functor between one-object categories $A$ and $B$) is an epimorphism in Cat.

12H. Prove that every functor that is full, faithful, and dense preserves and reflects initial objects, terminal objects, projective objects, injective objects, separators, and coseparators.

12I. Show that a full, faithful, and dense functor does not necessarily reflect identities, but that every embedding does reflect identities.

12J. Show that a $\mathscr{C}$-object $A$ is a coseparator for $\mathscr{C}$ if and only if $hom(\_\_, A)$ is faithful.

12K. Generalize the notions of projective and injective objects and separators and coseparators to quasicategories.

12L. *Concretizable Categories and Subcategories of* Set
A category $\mathscr{A}$ is said to be concretizable provided that there is some faithful functor $U: \mathscr{A} \to$ Set (i.e., provided that for some $U$, $(\mathscr{A}, U)$ is a concrete category). Prove the following:

(a) Every subcategory of a concretizable category is concretizable.
(b) Every category with a separator is concretizable.
(c) Cat is concretizable.
(d) The category of sets and relations between them is concretizable.
(e) If the category $\mathscr{C}$ is concretizable, then so is the arrow category $\mathscr{C}^2$.
(f) If the category $\mathscr{C}$ is concretizable, then so is $\mathscr{C}^{op}$. [The contravariant power set functor $\mathscr{P}$: Set$^{op} \to$ Set is faithful.]
(g) Every small category is concretizable. [If $\mathscr{C}$ is small, for each $A \in Ob(\mathscr{C})$, let

$$U(A) = \bigcup \{hom_{\mathscr{C}}(B, A) \mid B \in Ob(\mathscr{C})\}$$

and if $f \in Mor(\mathscr{C})$, let $U(f)$ be defined by $U(f)(g) = f \circ g$.]
(h) Not every category is concretizable. [Let $\mathcal{O}$ be the class of all ordinal numbers, for each $\alpha \in \mathcal{O}$ let $B_\alpha$ and $C_\alpha$ be disjoint non-empty sets and let $A$ be a set disjoint from each $B_\alpha$ and each $C_\alpha$. Let

$$Ob(\mathscr{C}) = \{B_\alpha \mid \alpha \in \mathcal{O}\} \cup \{A\} \cup \{C_\alpha \mid \alpha \in \mathcal{O}\}.$$

Let

$$hom_{\mathscr{C}}(X, X) = \{1_X\} \text{ for each } X \in Ob(\mathscr{C});$$

$$hom_{\mathscr{C}}(A, B_\alpha) = \varnothing = hom_{\mathscr{C}}(C_\alpha, A) \text{ for all } \alpha \in \mathcal{O};$$

$$hom_{\mathscr{C}}(B_\alpha, B_\beta) = \varnothing = hom_{\mathscr{C}}(C_\alpha, C_\beta) \text{ if } \alpha \neq \beta;$$

$$hom_{\mathscr{C}}(B_\alpha, A) = \{f_\alpha\};$$

$$hom_{\mathscr{C}}(A, C_\alpha) = \{h_\alpha, g_\alpha\} \ (h_\alpha \neq g_\alpha);$$

$$hom_{\mathscr{C}}(B_\alpha, C_\alpha) = \{h_\alpha \circ f_\alpha, g_\alpha \circ f_\alpha\} \ (h_\alpha \circ f_\alpha \neq g_\alpha \circ f_\alpha);$$

$$hom_{\mathscr{C}}(B_\alpha, C_\beta) = \{g_\beta \circ f_\alpha\} = \{h_\beta \circ f_\alpha\} \text{ if } \alpha \neq \beta;$$

$$hom_{\mathscr{C}}(C_\alpha, B_\beta) = \varnothing \text{ for all } \alpha \in \mathcal{O}, \beta \in \mathcal{O}.$$

Verify that this determines a category $\mathscr{C}$ and that there is no faithful functor $U: \mathscr{C} \to$ **Set**.]

(i) A category $\mathscr{C}$ is concretizable if and only if it can be embedded in **Set**; i.e., if and only if there is some embedding functor $E: \mathscr{C} \to$ **Set**.

## §13 NATURAL TRANSFORMATIONS AND NATURAL ISOMORPHISMS

Until now we have seen morphisms as a way of "getting from one object to another" and functors as a way of "getting from one category to another". We are now able to define natural transformations as a way of "getting from one functor to another". In other words, functors can be regarded as morphisms between categories (as has been made precise in §11) and natural transformations can be regarded as morphisms between functors (as will be made precise in Proposition 13.7). Recall that a motivation for the concept of natural transformations has already been given in Chapter I, where it was shown that there is a "natural way" to go from the identity functor on the category of finite dimensional vector spaces over a field $F$ to the second dual functor on the same category, but that there is no "natural way" to go from the identity functor to the first dual functor. Historically, category theory was developed in order to study these and similar situations; i.e., to adequately deal with natural transformations.

**13.1 DEFINITION**

Let $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{A} \to \mathscr{B}$ be functors.

(1) A **natural transformation** (or **functor morphism**) from $F$ to $G$ is a triple $(F, \eta, G)$ where $\eta: Ob(\mathscr{A}) \to Mor(\mathscr{B})$ is a function satisfying the following conditions:

(i) For each $\mathscr{A}$-object $A$, $\eta(A)$ (usually denoted by $\eta_A$) is a $\mathscr{B}$-morphism $\eta_A: F(A) \to G(A)$.

(ii) For each $\mathscr{A}$-morphism $A \xrightarrow{f} A'$, the diagram

$$\begin{array}{ccccc} F(A) & \xrightarrow{\eta_A} & G(A) & \qquad & A \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle f} \\ F(A') & \xrightarrow[\eta_{A'}]{} & G(A') & & A' \end{array}$$

commutes.

(2) A natural transformation $(F, \eta, G)$ is called a **natural isomorphism** provided that for each $\mathscr{A}$-object $A$, $\eta_A$ is a $\mathscr{B}$-isomorphism.

(3) $F$ and $G$ are said to be **naturally isomorphic** (denoted by $F \cong G$) if and only if there exists a natural isomorphism from $F$ to $G$.

If $(F, \eta, G)$ is a natural transformation, then $F$ (resp. $G$) is sometimes called the **domain** (resp. **codomain**) of $(F, \eta, G)$. Analogously to the situation with functors, the notations $(F, \eta, G)$, $\eta: F \to G$, $F \xrightarrow{\eta} G$ and $(\eta_A): F \to G$ are used interchangeably and by abuse of notation one usually does not distinguish between $\eta$ and the triple $(F, \eta, G)$.

13.2 EXAMPLES

(1) The identity natural transformation $1_F$ from any functor $F$ to itself, which assigns to each object $A$ the identity morphism on $F(A)$, is a natural isomorphism.

(2) For each field $F$, there is a natural transformation $\eta = (\eta_A)$ from the identity functor on $F$-**Mod** to the second dual functor, which assigns to each vector space $A$ the linear transformation $\eta_A$ defined by:

$$(\eta_A(x))(g) = g(x).$$

Note that the second dual functor is precisely

$$Hom(\_\_, F) \circ Hom(\_\_, F)^{op},$$

where

$$Hom(\_\_, F): (F\text{-}\mathbf{Mod})^{op} \to F\text{-}\mathbf{Mod}.$$

In exactly the same way, one can define a natural transformation from the identity functor on the category of normed linear spaces to the second dual functor and from the identity functor on the category of locally compact abelian groups to the second dual functor. The latter is a natural isomorphism.

(3) Let $2 = \{0, 1\}$ be a two-element set. Then there is a natural isomorphism $\eta = (\eta_A)$ from the contravariant *hom*-functor $hom(\_\_, 2)$ on **Set** to the contravariant power set functor $\mathscr{P}$, defined by:

$$\eta_A(g) = g^{-1}[\{0\}].$$

(4) For each set $A$, there is a natural isomorphism $\eta = (\eta_B)$ from the "left product by $A$" functor $(A \times \_\_)$: **Set** $\to$ **Set** to the "right product by $A$" functor $(\_\_ \times A)$: **Set** $\to$ **Set**, defined by:

$$\eta_B((a, b)) = (b, a).$$

(5) There is a natural transformation from the commutator subgroup functor $F$: **Grp** $\to$ **Grp** (9.2(8)) to the identity functor $1_{\mathbf{Grp}}$, whose value at each group A is the inclusion homomorphism

$$A' = F(A) \hookrightarrow A.$$

(6) Let $H$: **Grp** $\to$ **Ab** be the abelianization functor (9.2(8)) and $K$: **Ab** $\to$ **Grp** the inclusion functor. There is a natural transformation from $1_{\mathbf{Grp}}$ to $K \circ H$, whose value at each group $A$ is the canonical homomorphism

$$A \xrightarrow{\natural} A/A'$$

(where $A'$ is the commutator subgroup of $A$).

(7) There is a natural transformation from the $n$th homology functor $H_n$ to the $(n - 1)$st homology functor $H_{n-1}$, which assigns to $A$ the connecting homomorphism $\delta_A: H_n(A) \to H_{n-1}(A)$.
(8) The assignment of the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ to each topological space $X$ is a natural transformation from the $n$th homotopy functor $\pi_n$: **Top** $\to$ **Grp** to the $n$th homology functor $H_n$: **Top** $\to$ **Grp**.
(9) If $\mathscr{C}$ is any category and $B \xrightarrow{f} C$ is a $\mathscr{C}$-morphism, then there is a natural transformation

$$\eta: hom_{\mathscr{C}}(C, \_\_) \to hom_{\mathscr{C}}(B, \_\_)$$

defined by

$$\eta_A(g) = g \circ f.$$

(10) Let $U$ be the underlying functor from **Grp** to **Set** and let $F$ be the free functor from **Set** to **Grp**. There exist natural transformations

$$\eta = (\eta_A): 1_{\mathbf{Set}} \to U \circ F$$

and

$$\varepsilon = (\varepsilon_B): F \circ U \to 1_{\mathbf{Grp}}$$

where $\eta_A: A \to U(F(A))$ is the insertion of the generators and $\varepsilon_B: F(U(B)) \to B$ is the unique group homomorphism induced by the identity function on $U(B)$.
(11) Let $\beta$: $\mathbf{CRegT_2} \to \mathbf{CompT_2}$ be the Stone-Čech compactification functor and let $T$: $\mathbf{CompT_2} \hookrightarrow \mathbf{CRegT_2}$ be the inclusion functor. There exist natural transformations

$$\eta = (\eta_X): 1_{\mathbf{CRegT_2}} \to T \circ \beta$$

and

$$\varepsilon = (\varepsilon_Y): \beta \circ T \to 1_{\mathbf{CompT_2}}$$

where for each space $X$, $\eta_X$ is the usual embedding of $X$ into its compactification, and for each $Y$, $\varepsilon_Y$ is the unique homeomorphism induced by the identity on $Y$.
(12) For each set $A$, there is a natural transformation

$$\eta = (\eta_B): (\_\_ \times A) \circ hom_{\mathbf{Set}}(A, \_\_) \to 1_{\mathbf{Set}}$$

where

$$\eta_B: B^A \times A \to B$$

is defined by

$$\eta_B(f, a) = f(a).$$

$\eta$ is called the **evaluation natural transformation.**
(13) For each set $A$, there is a natural isomorphism $\eta$ from the bifunctor

$$hom \circ ((\_\_ \times A)^{op} \times 1_{\mathbf{Set}}): \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$$

to the bifunctor

$$hom \circ (1_{\mathbf{Set}^{op}} \times hom(A, \_)) \colon \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$$

where

$$\eta_{BC} \colon C^{B \times A} \to (C^A)^B$$

is defined by

$$(\eta_{BC}(f)(b))(a) = f(b, a).$$

(14) For each abelian group $A$, there is a natural isomorphism $\eta$ from the bifunctor

$$Hom \circ ((\_ \otimes A)^{op} \times 1_{\mathbf{Ab}}) \colon \mathbf{Ab}^{op} \times \mathbf{Ab} \to \mathbf{Ab}$$

to the bifunctor

$$Hom \circ (1_{\mathbf{Ab}^{op}} \times Hom(A, \_)) \colon \mathbf{Ab}^{op} \times \mathbf{Ab} \to \mathbf{Ab}$$

where

$$\eta_{BC} \colon Hom((B \otimes A), C) \to Hom(B, Hom(A, C))$$

is defined by

$$(\eta_{BC}(f)(b))(a) = f(b \otimes a)$$

**13.3 DEFINITION**

The **composition of natural transformations** $F \xrightarrow{\eta} G$ and $G \xrightarrow{\varepsilon} H$ is the triple $(F, \xi, H)$ where $\xi$ is that function that assigns to each object $A$, the morphism $F(A) \xrightarrow{\varepsilon_A \cdot \eta_A} H(A)$. The composition is usually denoted by $F \xrightarrow{\varepsilon \circ \eta} H$.

**13.4 PROPOSITION**

*The composition of natural transformations (resp. natural isomorphisms) is a natural transformation (resp. natural isomorphism).*

*Proof:* This follows immediately from the commutativity of the diagram:

$$\begin{array}{ccccc} F(A) & \xrightarrow{\eta_A} & G(A) & \xrightarrow{\varepsilon_A} & H(A) \\ {\scriptstyle F(f)}\downarrow & & {\scriptstyle G(f)}\downarrow & & {\scriptstyle H(f)}\downarrow \\ F(B) & \xrightarrow{\eta_B} & G(B) & \xrightarrow{\varepsilon_B} & H(B) \end{array}$$

for each $A \xrightarrow{f} B$ in the domain of $F$, $G$, and $H$, and from the fact that the composition of isomorphisms is an isomorphism. □

**13.5 PROPOSITION**

*The composition of natural transformations is associative; i.e., if $\delta$, $\eta$, and $\varepsilon$ are natural transformations such that $\delta \circ \eta$ and $\eta \circ \varepsilon$ are defined, then $(\delta \circ \eta) \circ \varepsilon$ and $\delta \circ (\eta \circ \varepsilon)$ are defined and are equal.* □

**13.6 THEOREM**

*A natural transformation $\eta: F \to G$ is a natural isomorphism if and only if there exists some natural transformation $\delta: G \to F$ such that $\delta \circ \eta = 1_F$ and $\eta \circ \delta = 1_G$.*

*Proof:* Assume that $\mathscr{A} \xrightarrow{F} \mathscr{B}$ and $\mathscr{A} \xrightarrow{G} \mathscr{B}$.

(1) If $\eta: F \to G$ is a natural isomorphism, define the function

$$\delta: Ob(\mathscr{A}) \to Mor(\mathscr{B}) \quad \text{by} \quad \delta(A) = \delta_A = \eta_A^{-1}.$$

Clearly, $\delta: G \to F$ is a natural transformation, $\delta \circ \eta = 1_F$ and $\eta \circ \delta = 1_G$.
(2) If $\delta \circ \eta = 1_F$ and $\eta \circ \delta = 1_G$, then for each $A \in Ob(\mathscr{A})$, $\delta_A \circ \eta_A = (\delta \circ \eta)_A = 1_{F(A)}$ and $\eta_A \circ \delta_A = (\eta \circ \delta)_A = 1_{G(A)}$. Hence, each $\eta_A$ is an isomorphism. □

**13.7 PROPOSITION**

*There exists a quasicategory $(\mathfrak{F}, \mathfrak{N}, dom, cod, \circ)$ where $\mathfrak{F}$ is the conglomerate of all functors, $\mathfrak{N}$ is the conglomerate of all natural transformations, dom and cod are functions that assign to each natural transformation its domain and codomain, respectively, and $\circ$ is the composition of natural transformations defined above* (13.3). □

**13.8 DEFINITION**

The quasicategory described in the above proposition is called the **quasicategory of all functors** and is denoted by $\mathscr{FUNC}$.

Analogously to the situation with $\mathscr{CAT}$, when we restrict our attention to functors between small categories and natural transformations between them, we obtain a category, denoted by **Func** (see Exercise 13H).

### Star Products

We next investigate another way of composing natural transformations.

**13.9 PROPOSITION**

*Let $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ and $\mathscr{B} \underset{K}{\overset{H}{\rightrightarrows}} \mathscr{C}$ be functors and let $\eta: F \to G$ and $\delta: H \to K$ be natural transformations. Then for each $A \in Ob(\mathscr{A})$, the square*

$$\begin{array}{ccc} (H \circ F)(A) & \xrightarrow{H(\eta_A)} & (H \circ G)(A) \\ \downarrow{\scriptstyle \delta_{FA}} & & \downarrow{\scriptstyle \delta_{GA}} \\ (K \circ F)(A) & \xrightarrow[K(\eta_A)]{} & (K \circ G)(A) \end{array}$$

*commutes.*

*Furthermore, if $\mu: Ob(\mathscr{A}) \to Mor(\mathscr{C})$ is the function that sends each $\mathscr{A}$-object $A$ to the diagonal of the above square, i.e.,*

$$\mu_A = \delta_{GA} \circ H(\eta_A) = K(\eta_A) \circ \delta_{FA},$$

*then $(H \circ F, \mu, K \circ G)$ is a natural transformation.*

*Proof:* The square commutes because $\eta_A: F(A) \to G(A)$ is a $\mathscr{B}$-morphism and $\delta$ is a natural transformation from $H$ to $K$.

To see that $\mu$ is a natural transformation, let $A \xrightarrow{f} A'$ be an $\mathscr{A}$-morphism. Since $\eta$ is a natural transformation, the square

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow[\eta_{A'}]{} & G(A') \end{array} \qquad \begin{array}{c} A \\ \downarrow{\scriptstyle f} \\ A' \end{array}$$

commutes.

Applying $H$ to this, we see that the left square of the diagram

$$\begin{array}{ccccc} & & \overset{\mu_A}{\frown\!\!\!\longrightarrow} & & \\ (H\circ F)(A) & \xrightarrow{H(\eta_A)} & (H\circ G)(A) & \xrightarrow{\delta_{GA}} & (K\circ G)(A) \\ {\scriptstyle (H\circ F)(f)}\downarrow & & \downarrow{\scriptstyle (H\circ G)(f)} & & \downarrow{\scriptstyle (K\circ G)(f)} \\ (H\circ F)(A') & \xrightarrow[H(\eta_{A'})]{} & (H\circ G)(A') & \xrightarrow[\delta_{GA'}]{} & (K\circ G)(A') \\ & & \underset{\mu_{A'}}{\smile\!\!\!\longrightarrow} & & \end{array}$$

commutes.

Since $G(f): G(A) \to G(A')$ is a $\mathscr{B}$-morphism, and since $\delta$ is a natural transformation from $H$ to $K$, the right square commutes. Thus, $\mu$ is a natural transformation from $H \circ F$ to $K \circ G$. □

**13.10 DEFINITION**

The natural transformation $H \circ F \xrightarrow{\mu} K \circ G$ constructed in Proposition 13.9 is called the **star product** of $\delta$ and $\eta$ and is denoted by $\delta * \eta$.

**13.11 PROPOSITION**

*The star product is associative; i.e., whenever $\delta * \eta$ and $\eta * \varepsilon$ are defined, then $(\delta * \eta) * \varepsilon$ and $\delta * (\eta * \varepsilon)$ are defined and are equal.*

*Proof:* Either side is defined if and only if we have a situation such as:

$$\begin{array}{ccccccc} \mathscr{A} & \xrightarrow{S} & \mathscr{B} & \xrightarrow{F} & \mathscr{C} & \xrightarrow{H} & \mathscr{D} \\ & {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \delta} & \\ \mathscr{A} & \xrightarrow[T]{} & \mathscr{B} & \xrightarrow[G]{} & \mathscr{C} & \xrightarrow[K]{} & \mathscr{D} \end{array}$$

Applying the definition of star product several times, we obtain

$$\begin{aligned} ((\delta * \eta) * \varepsilon)_A &= (K \circ G)(\varepsilon_A) \circ (\delta * \eta)_{SA} = K(G(\varepsilon_A)) \circ (K(\eta_{SA}) \circ \delta_{(F\circ S)A}) \\ &= K(G(\varepsilon_A) \circ \eta_{SA}) \circ \delta_{(F\circ S)A} = (\delta * (\eta * \varepsilon))_A. \quad \square \end{aligned}$$

**13.12 THEOREM (INTERCHANGE LAW)**

*For any natural transformations* $\nu$, $\mu$, $\eta$, *and* $\varepsilon$

$$(\nu \circ \mu) * (\eta \circ \varepsilon) = (\nu * \eta) \circ (\mu * \varepsilon);$$

*i.e., when the left side is defined, then so is the right side and they are equal.*

*Proof:* For the left side to be defined, we must have a situation such as:

$$\begin{array}{ccccc}
\mathscr{A} & \xrightarrow{F} & \mathscr{B} & \xrightarrow{K} & \mathscr{C} \\
 & \varepsilon\downarrow & & \downarrow\mu & \\
\mathscr{A} & \xrightarrow{G} & \mathscr{B} & \xrightarrow{L} & \mathscr{C} \\
 & \eta\downarrow & & \downarrow\nu & \\
\mathscr{A} & \xrightarrow{H} & \mathscr{B} & \xrightarrow{M} & \mathscr{C}
\end{array}$$

Thus, if the left side is defined, then so is the right. By the definition of star product and composition of natural transformations,

$$((\nu \circ \mu) * (\eta \circ \varepsilon))_A = (\nu \circ \mu)_{HA} \circ K((\eta \circ \varepsilon)_A) = \nu_{HA} \circ \mu_{HA} \circ K(\eta_A) \circ K(\varepsilon_A).$$

But since $\mu$ is a natural transformation and $G(A) \xrightarrow{\eta_A} H(A)$ is a morphism in $\mathscr{B}$, the square

$$\begin{array}{ccc}
K(G(A)) & \xrightarrow{K(\eta_A)} & K(H(A)) \\
\mu_{G(A)}\downarrow & & \downarrow\mu_{H(A)} \\
L(G(A)) & \xrightarrow[L(\eta_A)]{} & L(H(A))
\end{array}$$

commutes.

Thus, the two middle terms of the above expression can be replaced to obtain

$$\nu_{H(A)} \circ L(\eta_A) \circ \mu_{G(A)} \circ K(\varepsilon_A).$$

But by the definition of star product, this is

$$((\nu * \eta) \circ (\mu * \varepsilon))_A. \quad \square$$

**13.13** For typographical reasons, instead of writing $1_F$ for the identity natural transformation on the functor $F$, we often use the symbol "$F$" itself. Thus, for example,

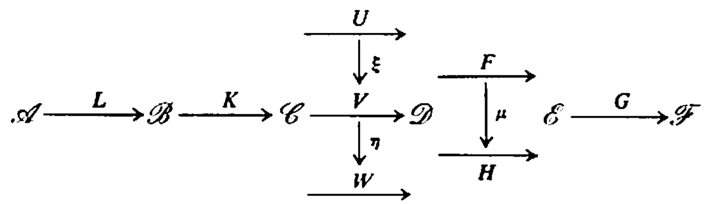
$$(\eta * F)_A = \eta_{F(A)}$$

and

$$(F * \varepsilon)_A = F(\varepsilon_A).$$

**13.14 COROLLARY (GODEMENT'S "FIVE" RULES)**

*Given the functors and natural transformations,*



*the following hold:*

(1) $(G \circ F) * \xi = G * (F * \xi)$

(2) $\xi * (K \circ L) = (\xi * K) * L$

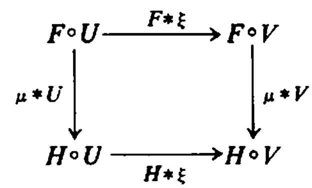
(3) $1_U * K = 1_{U \circ K}$

(4) $F * 1_U = 1_{F \circ U}$

(5) $F * (\xi * K) = (F * \xi) * K$

(6) $F * (\eta \circ \xi) * K = (F * \eta * K) \circ (F * \xi * K)$

(7) *The square*



*commutes.* □

**13.15 PROPOSITION**

*There exists a quasicategory* ($\mathfrak{C}$, $\mathfrak{N}$, *dom*, *cod*, $*$) *where* $\mathfrak{C}$ *is the conglomerate of all categories,* $\mathfrak{N}$ *is the conglomerate of all natural transformations, dom is the function that assigns to each natural transformation the category that is the domain of its domain, cod is the function that assigns to each natural transformation the codomain of its domain, and* $*$ *is the star product of natural transformations.* □

**13.16** The above proposition together with Proposition 13.7 shows that the conglomerate $\mathfrak{N}$ of all natural transformations can be thought of as the morphisms of a quasicategory in two different ways; i.e., with the "usual composition" $\circ$ or with the "star product" $*$. Also, these two types of composition are intimately linked by the "interchange law" (13.12). Thus, the triple ($\mathfrak{N}$, $\circ$, $*$) is sometimes called the "**double quasicategory**", $\mathcal{NAT}$, **of natural transformations.**

## EXERCISES

13A. Show that in general none of the natural transformations of Example 13.2(5), (6), (7), (8), (9), (10), (11), or (12) is a natural isomorphism.

13B. Prove that the trifunctors of Exercise 10A are naturally isomorphic.

13C. Show that if $(G, \eta, F)$ is a natural transformation (resp. natural isomorphism), then $(F^{op}, \eta, G^{op})$ is a natural transformation (resp. natural isomorphism). [$(F^{op}, \eta, G^{op})$ is sometimes called the *opposite* of $(G, \eta, F)$.]

13D. Let $2 = \{0, 1\}$ be a two-element set. Prove that there is a natural isomorphism between the covariant *hom*-functor $hom(2, \_)$ on **Set** and the squaring functor $(\quad)^2$ (see Exercise 9A(b)).

13E. Show that if $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ are functors and $\eta: F \to G$ is a natural transformation, then $\eta$ can be regarded as a functor from $\mathscr{A}$ to $\mathscr{B}^2$ (where $\mathscr{B}^2$ is the arrow-category for $\mathscr{B}$ (4.16)).

13F. Show that there are at least two natural transformations from the "free group" functor $F$: **Set** $\to$ **Grp** to the "free abelian group" functor $G$: **Set** $\to$ **Grp**.

13G. Let

$$Hom_{\mathbf{BooSp}}(\text{—}, 2): \mathbf{BooSp}^{op} \to \mathbf{BooAlg}$$

and

$$Hom_{\mathbf{BooAlg}}(\text{—}, 2): \mathbf{BooAlg} \to \mathbf{BooSp}^{op}$$

(see Examples 10.6(7) and (8)). Show that

$$Hom_{\mathbf{BooSp}}(\text{—}, 2) \circ Hom_{\mathbf{BooAlg}}(\text{—}, 2) \cong 1_{\mathbf{BooAlg}}$$

and

$$Hom_{\mathbf{BooAlg}}(\text{—}, 2) \circ Hom_{\mathbf{BooSp}}(\text{—}, 2) \cong 1_{\mathbf{BooSp}^{op}}.$$

13H. Let **Func** be the full subquasicategory of $\mathscr{FUNC}$ whose objects are the functors between small categories. Prove that **Func** is a category.

13I. In Example 13.2(10), show that

$$(U * \varepsilon) \circ (\eta * U) = 1_U$$

and that

$$(\varepsilon * F) \circ (F * \eta) = 1_F.$$

13J. In Example 13.2(11), show that

$$(T * \varepsilon) \circ (\eta * T) = 1_T$$

and that

$$(\varepsilon * \beta) \circ (\beta * \eta) = 1_\beta.$$

13K. Show that if $F, G: \mathscr{A} \to \mathscr{B}$ are functors and $\mathscr{B}$ is pointed, then there exists a natural transformation $\eta: F \to G$.

13L. Prove that a group endomorphism is an inner automorphism if and only if, considered as a functor on a one-object category, it is naturally isomorphic to the identity functor.

## §14 ISOMORPHISMS AND EQUIVALENCES OF CATEGORIES

In this section, we consider the problem of determining what it means for two categories to be "essentially the same". We begin by introducing the notion of "isomorphism" of categories, which seems appropriate at first glance, but which is too strong. The weaker concept of "equivalence" of categories is shown to be the proper notion for "essential sameness".

### 14.1 DEFINITION

(1) A functor $F: \mathscr{A} \to \mathscr{B}$ is said to be an **isomorphism from $\mathscr{A}$ to $\mathscr{B}$** provided that it is an isomorphism in the quasicategory $\mathscr{CAT}$; i.e., provided that there exists a functor $G: \mathscr{B} \to \mathscr{A}$ such that $G \circ F = 1_{\mathscr{A}}$ and $F \circ G = 1_{\mathscr{B}}$.

(2) Categories $\mathscr{A}$ and $\mathscr{B}$ are said to be **isomorphic** (denoted by $\mathscr{A} \cong \mathscr{B}$) provided that there is an isomorphism between them.

### 14.2 EXAMPLES

(1) Every identity functor is an isomorphism.

(2) There is an isomorphism from **Rng** to itself which sends each ring $R$ to its opposite ring $R^*$.

(3) For every ring $R$, $R$-**Mod** is isomorphic with **Mod**-$R^*$.

(4) The category **Rng** is isomorphic with the category **Z-Alg** (where **Z** is the ring of integers).

(5) **Ab** and **Z-Mod** are isomorphic.

(6) The category **BooAlg** is isomorphic with the category of boolean rings† together with ring homomorphisms.

(7) For any category $\mathscr{C}$, $(\mathscr{C} \times \mathscr{C})^{op} \cong \mathscr{C}^{op} \times \mathscr{C}^{op}$.

(8) Let $A$ be a one-element (resp. two-element) set. Then the comma category $(A, \mathbf{Set})$ is isomorphic with the category of pointed sets (resp. the category of bi-pointed sets) (see Exercise 4S).

(9) $\mathbf{TopBun} \cong \mathbf{Top}^2$, and for any topological space $B$, $\mathbf{TopBun}_B \cong (\mathbf{Top}, B)$ (see Exercise 4S).

(10) A category is concretizable if and only if it is isomorphic with some subcategory of **Set** (see Exercise 12L).

### 14.3 PROPOSITION (CHARACTERIZATION OF ISOMORPHISMS)

*If $F: \mathscr{A} \to \mathscr{B}$ is a functor, then the following are equivalent:*

(1) *$F$ is an isomorphism.*

(2) *The function $F$: $Mor(\mathscr{A}) \to Mor(\mathscr{B})$ is a bijection.*

(3) *$F$ is full and faithful and the associated object function $F$: $Ob(\mathscr{A}) \to Ob(\mathscr{B})$ is a bijection.*

† A **boolean ring** is a ring in which each element is idempotent with respect to multiplication.

*Proof:* The equivalence of (1) and (2) follows from the fact that for any category $\mathscr{C}$, the identity function on $Mor(\mathscr{C})$ is the identity functor on $\mathscr{C}$. The equivalence of (2) and (3) follows from the one-to-one correspondence between objects and identities in any category. □

The category $F$-**Mod** of all vector spaces over a field $F$ and its full subcategory $\mathscr{C}$ consisting of all objects of the form $F^I$ (i.e., all powers of the field $F$) are obviously not the same nor even isomorphic. Yet, from a categorical point of view, they are "essentially the same", i.e., they have the same categorical characteristics. The main difference between the two lies in the fact that in $\mathscr{C}$ any two isomorphic objects are identical, whereas there exist many different objects isomorphic to any given object in $F$-**Mod**. Below we will define "equivalence of categories" in such a way that two categories $\mathscr{A}$ and $\mathscr{B}$ will be equivalent provided that the only difference between them lies in the fact that in one of them some objects might be "counted" more times than in the other—in other words, provided that the categories obtained from $\mathscr{A}$ and $\mathscr{B}$ by "counting" each object just once, are isomorphic.

**14.4 DEFINITION**

(1) A category $\mathscr{C}$ is called **skeletal** provided that isomorphic $\mathscr{C}$-objects are identical.

(2) A **skeleton** of a category $\mathscr{C}$ is a maximal full skeletal subcategory of $\mathscr{C}$.

**14.5 EXAMPLES**

(1) The full subcategory of all cardinal numbers is a skeleton for **Set**.

(2) For any field $F$, the full subcategory of all powers $F^I$ is a skeleton for $F$-**Mod**.

(3) For any field $F$, the full subcategory of all finite powers $F^n$ is a skeleton for the category of all finite-dimensional vector spaces over $F$.

**14.6 PROPOSITION**

*Every category $\mathscr{C}$ has a skeleton.*

*Proof:* If we let $A \cong B$ mean that there is a $\mathscr{C}$-isomorphism from $A$ to $B$, then $\cong$ is an equivalence relation on $Ob(\mathscr{C})$. Hence, by the Axiom of Choice (1.2(4)), it has a system of representatives $\mathscr{A}$. Let $\mathscr{B}$ be the full subcategory of $\mathscr{C}$ that is generated by $\mathscr{A}$. Clearly, $\mathscr{B}$ is skeletal and is contained in no other full skeletal subcategory of $\mathscr{C}$. □

**14.7 PROPOSITION**

*Any two skeletons of a category are isomorphic.* □

**14.8 DEFINITION**

A category $\mathscr{A}$ is said to be **equivalent** to a category $\mathscr{B}$ (denoted by $\mathscr{A} \sim \mathscr{B}$) if and only if $\mathscr{A}$ and $\mathscr{B}$ have isomorphic skeletons.

**14.9 PROPOSITION**

*The relation "is equivalent to" is an equivalence relation on the conglomerate of all categories.* □

**14.10 PROPOSITION**

*Skeletal categories are equivalent if and only if they are isomorphic.* □

Categories $\mathscr{A}$ and $\mathscr{B}$ have been defined to be isomorphic provided that there is an isomorphism $F: \mathscr{A} \to \mathscr{B}$. Likewise, we will define special functors called "equivalences" in such a way that $\mathscr{A}$ and $\mathscr{B}$ will be equivalent provided that there is an equivalence $F: \mathscr{A} \to \mathscr{B}$.

**14.11 THEOREM**

*If $F: \mathscr{A} \to \mathscr{B}$ is a functor, then the following are equivalent:*

(1) *$F$ is full, faithful, and dense.*

(2) *There is a functor $G: \mathscr{B} \to \mathscr{A}$ such that $F \circ G \cong 1_{\mathscr{B}}$ and $G \circ F \cong 1_{\mathscr{A}}$.*

(3) *There is a functor $G: \mathscr{B} \to \mathscr{A}$ and natural isomorphisms $\eta: 1_{\mathscr{A}} \to G \circ F$ and $\varepsilon: F \circ G \to 1_{\mathscr{B}}$ such that $F * \eta = (\varepsilon * F)^{-1}$ and $G * \varepsilon = (\eta * G)^{-1}$.*

*Proof:* Clearly (3) implies (2). We will show that (2) implies (1) and (1) implies (3).

*(2) ⇒ (1).* Suppose that $G: \mathscr{B} \to \mathscr{A}$ such that $G \circ F \cong 1_{\mathscr{A}}$ and $F \circ G \cong 1_{\mathscr{B}}$. Let $\eta: 1_{\mathscr{A}} \to G \circ F$ be a natural isomorphism.

(i) If $A \overset{f}{\underset{g}{\rightrightarrows}} A'$ are $\mathscr{A}$-morphisms such that $F(f) = F(g)$, and if $A \dashrightarrow A'$ denotes either $f$ or $g$, then the square

$$\begin{array}{ccc} A & \xrightarrow{\ \eta_A\ } & G \circ F(A) \\ \vdots\!\downarrow & & \downarrow {\scriptstyle G \circ F(f) = G \circ F(g)} \\ A' & \xrightarrow[\ \eta_{A'}\ ]{} & G \circ F(A') \end{array}$$

commutes. Thus, $\eta_{A'} \circ f = \eta_{A'} \circ g$ so that (since $\eta_{A'}$ is a monomorphism) $f = g$. Thus, $F$ is faithful. Likewise, $G$ is faithful.

(ii) Since $F \circ G \cong 1_{\mathscr{B}}$, we know that for each $B \in Ob(\mathscr{B})$,

$$F \circ G(B) \cong 1_{\mathscr{B}}(B) = B.$$

Thus, $F$ is dense.

(iii) Suppose that $F(A) \xrightarrow{g} F(A')$. Then $f = \eta_{A'}^{-1} \circ G(g) \circ \eta_A$ is a morphism from $A$ to $A'$ such that the square

$$\begin{array}{ccc} A & \xrightarrow{\ \eta_A\ } & G \circ F(A) \\ {\scriptstyle f}\downarrow & & \downarrow {\scriptstyle G(g)} \quad \text{or} \quad G \circ F(f) \\ A' & \xrightarrow[\ \eta_{A'}\ ]{} & G \circ F(A') \end{array}$$

commutes. Thus, $G(g) \circ \eta_A = G \circ F(f) \circ \eta_A$ so that (since $\eta_A$ is an epimorphism) $G(g) = G \circ F(f)$. Since $G$ is faithful, $g = F(f)$. Consequently, $F$ is full.

(*1*) $\Rightarrow$ (*3*). Let $\mathscr{A}$ be a system of representatives for the equivalence relation $\cong$ on $Ob(\mathscr{A})$. [Such exists by the Axiom of Choice (1.2(4)).] Since $F$ is dense and since full and faithful functors preserve and reflect isomorphisms, for each $B \in Ob(\mathscr{B})$ there is a unique member of $\mathscr{A}$ (which we will denote by $G(B)$) such that $F(G(B)) \cong B$. Thus, $G\colon Ob(\mathscr{B}) \to Ob(\mathscr{A})$ is well-defined. Again using the Axiom of Choice, for each $B \in Ob(\mathscr{B})$ choose one isomorphism $\varepsilon_B\colon F(G(B)) \to B$.

We will now define $G$ on the morphisms of $\mathscr{B}$. If $B \xrightarrow{g} B'$ is a $\mathscr{B}$-morphism, then the square

$$\begin{array}{ccc} F(G(B)) & \xrightarrow{\varepsilon_B} & B \\ {\scriptstyle \varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B}\downarrow & & \downarrow{\scriptstyle g} \\ F(G(B')) & \xrightarrow[\varepsilon_{B'}]{} & B' \end{array}$$

commutes. Since $F$ is full and faithful, there is a unique $\mathscr{A}$-morphism

$$f\colon G(B) \to G(B')$$

such that $F(f) = \varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B$. We let $G(g) = f$ and in this manner establish a well-defined function $G\colon Mor(\mathscr{B}) \to Mor(\mathscr{A})$. For any $B \in Ob(\mathscr{B})$,

$$F(G(1_B)) = \varepsilon_B^{-1} \circ 1_B \circ \varepsilon_B = 1_{F(G(B))} = F(1_{G(B)}).$$

So that, since $F$ is faithful, $G(1_B) = 1_{G(B)}$. Hence $G$ preserves identities.

If $B \xrightarrow{g} B' \xrightarrow{h} B''$, then

$$\begin{aligned} F(G(h \circ g)) &= \varepsilon_{B''}^{-1} \circ h \circ g \circ \varepsilon_B = (\varepsilon_{B''}^{-1} \circ h \circ \varepsilon_{B'}) \circ (\varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B) \\ &= F(G(h)) \circ F(G(g)) = F(G(h) \circ G(g)). \end{aligned}$$

Thus, since $F$ is faithful, $G(h \circ g) = G(h) \circ G(g)$, so that $G$ preserves composition. Consequently, $G$ is a functor. Clearly, by the definition of $G$, $\varepsilon = (\varepsilon_B)$ is a natural isomorphism from $F \circ G$ to $1_{\mathscr{B}}$.

To establish a natural isomorphism from $1_{\mathscr{A}}$ to $G \circ F$, note that for each $A \in Ob(\mathscr{A})$, $\varepsilon_{FA}\colon F \circ G \circ F(A) \to F(A)$ is a $\mathscr{B}$-isomorphism. Thus, since $F$ is full and faithful, there is a unique $\mathscr{A}$-isomorphism $\eta_A\colon A \to G \circ F(A)$ such that $F(\eta_A) = \varepsilon_{FA}^{-1}$ (12.9). Since $\varepsilon^{-1}$ is natural, for any $A \xrightarrow{f} A'$ the square

$$\begin{array}{ccc} F(A) & \xrightarrow{\varepsilon_{FA}^{-1} = F(\eta_A)} & F \circ G \circ F(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F \circ G \circ F(f)} \\ F(A') & \xrightarrow[\varepsilon_{FA'}^{-1} = F(\eta_{A'})]{} & F \circ G \circ F(A') \end{array}$$

commutes. Thus, since faithful functors reflect commutativity (12.8), the square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & G\circ F(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G\circ F(f)} \\ A' & \xrightarrow[\eta_{A'}]{} & G\circ F(A') \end{array}$$

commutes. Hence, $\eta = (\eta_A)$ is a natural isomorphism from $1_{\mathscr{A}}$ to $G \circ F$.

Clearly, by the definition of $\eta$, $F * \eta = (\varepsilon * F)^{-1}$. Also by the definition of $\eta$, $F(\eta_{G(B)}) = \varepsilon_{FG(B)}^{-1}$. Hence, $F(\eta_{G(B)}^{-1}) = \varepsilon_{FG(B)}$. By the definition of $G$, $F \circ G(\varepsilon_B) = \varepsilon_B^{-1} \circ \varepsilon_B \circ \varepsilon_{F \circ G(B)} = \varepsilon_{F \circ G(B)}$. Thus $F(\eta_{G(B)}^{-1}) = F(G(\varepsilon_B))$, so that since $F$ is faithful, $(\eta * G)^{-1}(B) = (G * \varepsilon)(B)$. □

**14.12 DEFINITION**

(1) A functor $F$ is called an **equivalence** provided that it fulfills the equivalent conditions of the above theorem, i.e., provided that it is full, faithful, and dense.
(2) $(F, G, \eta, \varepsilon)$ is called an **equivalence situation** provided that $F$ and $G$ are functors and $\eta: 1 \to G \circ F$ and $\varepsilon: F \circ G \to 1$ are natural isomorphisms such that $F * \eta = (\varepsilon * F)^{-1}$ and $G * \varepsilon = (\eta * G)^{-1}$.

**14.13 PROPOSITION**

(1) *The composition of equivalences is an equivalence.*

(2) *If $F$ is an equivalence, then so is $F^{op}$.*

(3) *If $(F, G, \eta, \varepsilon)$ is an equivalence situation, then so is $(G, F, \varepsilon^{-1}, \eta^{-1})$.*

*Proof:* (1) and (2) follow from the fact that each of the properties—fullness, faithfulness, and density—is closed under composition and is self-dual. (3) follows from the fact that for any isomorphism $f$, $(f^{-1})^{-1} = f$. □

**14.14 PROPOSITION**

*If $\mathscr{S}$ is a skeleton of a category $\mathscr{C}$ and $E: \mathscr{S} \hookrightarrow \mathscr{C}$ is the inclusion functor, then there exists a functor $P: \mathscr{C} \to \mathscr{S}$ (called the* **projection of $\mathscr{C}$ onto $\mathscr{S}$***) such that $P \circ E = 1_{\mathscr{S}}$ (i.e., $E$ is a section in $\mathscr{CAT}$). Furthermore, both $E$ and $P$ are equivalences.* □

**14.15 THEOREM**

*Categories $\mathscr{A}$ and $\mathscr{B}$ are equivalent if and only if there exists an equivalence $F: \mathscr{A} \to \mathscr{B}$.*

*Proof:* If $\mathscr{A} \sim \mathscr{B}$, then there exist skeletons $\hat{\mathscr{A}}$ and $\hat{\mathscr{B}}$ of $\mathscr{A}$ and $\mathscr{B}$, respectively, such that $\hat{\mathscr{A}} \cong \hat{\mathscr{B}}$. By the above proposition, the projection $P: \mathscr{A} \to \hat{\mathscr{A}}$ and the inclusion $E: \hat{\mathscr{B}} \hookrightarrow \mathscr{B}$ are equivalences. By definition, there exists an isomorphism $J: \hat{\mathscr{A}} \to \hat{\mathscr{B}}$ (which must be an equivalence). Thus $E \circ J \circ P: \mathscr{A} \to \mathscr{B}$ is an equivalence (14.13).

Conversely, if $F: \mathscr{A} \to \mathscr{B}$ is an equivalence, $\hat{\mathscr{A}}$ and $\hat{\mathscr{B}}$ are skeletons for $\mathscr{A}$ and $\mathscr{B}$, $\hat{\mathscr{A}} \overset{K}{\hookrightarrow} \mathscr{A}$ is the inclusion and $\mathscr{B} \xrightarrow{Q} \hat{\mathscr{B}}$ is the projection, then

$Q \circ F \circ K: \hat{\mathscr{A}} \to \hat{\mathscr{B}}$ is an equivalence. Thus, $Q \circ F \circ K$ is full and faithful, and since $\hat{\mathscr{A}}$ and $\hat{\mathscr{B}}$ are skeletal, the associated object function for $Q \circ F \circ K$ must be a bijection. Hence, it is an isomorphism (14.3), so that $\mathscr{A}$ and $\mathscr{B}$ are equivalent. □

**14.16 EXAMPLES OF EQUIVALENCES**

(1) Every isomorphism is an equivalence.

(2) Every inclusion of a skeleton and every projection onto a skeleton is an equivalence (14.14).

(3) If $F: \mathscr{C} \to \mathscr{C}$ is naturally isomorphic to $1_{\mathscr{C}}$, then $F$ is an equivalence, but not conversely (see Exercise 14D).

(4) For any field $F$, the category of finite dimensional vector spaces over $F$ is equivalent to the category of $F$-matrices (3.5(4)). [Given a basis, consider the matrix associated with each linear transformation.]

(5) For any field $F$, let $\mathscr{A}$ be the category of finite dimensional vector spaces over $F$. Then the functor $Hom(\_\_, F): \mathscr{A}^{op} \to \mathscr{A}$ that sends each vector space to its dual is an equivalence, but not an isomorphism.

(6) Let $\mathscr{L}$ be the category of reflexive locally convex Hausdorff linear topological spaces. The functor $Hom(\_\_, \mathbf{C}): \mathscr{L}^{op} \to \mathscr{L}$ that sends each such space to its adjoint (equipped with the strong topology), is an equivalence, but not an isomorphism.

(7) Let $\mathscr{R}$ be the category of reflexive Banach spaces and norm-decreasing linear transformations. The functor $Hom(\_\_, \mathbf{C}): \mathscr{R}^{op} \to \mathscr{R}$ that sends each reflexive Banach space to its conjugate space is an equivalence, but not an isomorphism.

(8) Let **LCAb** be the category of locally compact abelian groups. Then the functor $Hom(\_\_, \mathbf{R}/\mathbf{Z}): \mathbf{LCAb}^{op} \to \mathbf{LCAb}$ that sends each locally compact abelian group to its group of characters is an equivalence, but not an isomorphism.

(9) Let **CompAb** be the category of compact abelian groups. Then the functor $Hom(\_\_, \mathbf{R}/\mathbf{Z}): \mathbf{CompAb}^{op} \to \mathbf{Ab}$, that sends each compact abelian group to its (discrete) group of characters is an equivalence, but not an isomorphism.

(10) The functor $Hom(\_\_, 2): \mathbf{BooSp}^{op} \to \mathbf{BooAlg}$ that sends each boolean space to the boolean algebra of its clopen subsets is an equivalence, but not an isomorphism.

(11) The functor $Hom(\_\_, \mathbf{C}): \mathbf{C^*\text{-}Alg}^{op} \to \mathbf{CompT_2}$ that sends each $C^*$-algebra $A$ to its carrier space; i.e., $Hom(A, \mathbf{C})$ considered as a subspace of the space $\mathbf{C}^A$, is an equivalence, but not an isomorphism.

The preceding examples indicate quite strongly that the concept of equivalence is much more important categorically than the concept of isomorphism of categories. Moreover, since equivalences preserve (and reflect) all "essential" categorical properties (see Theorem 12.10 and Exercise 12H), one might even define a property of categories to be "categorical" provided that it is preserved by equivalences.

The last few of the preceding examples also motivate the following definition.

**14.17 DEFINITION**

(1) Categories $\mathscr{A}$ and $\mathscr{B}$ are called **dually equivalent** if and only if $\mathscr{A}^{op}$ and $\mathscr{B}$ are equivalent.

(2) A category $\mathscr{A}$ is called **self-dual** provided that it is dually equivalent to itself.

**14.18 EXAMPLES**

(1) For any category $\mathscr{C}$, $\mathscr{C}$ and $\mathscr{C}^{op}$ are dually equivalent.

(2) For any commutative ring $R$, the category of $R$-matrices (3.5(4)) is self-dual. [Consider the functor that sends each matrix to its transpose.]

(3) The category of finite dimensional vector spaces over any field is self-dual.

(4) The category of reflexive Banach spaces is self-dual.

(5) The category of locally compact abelian groups is self-dual.

(6) The category of compact abelian groups is dually equivalent to **Ab**.

(7) The category of boolean spaces is dually equivalent to **BooAlg**.

(8) $\mathbf{CompT_2}$ is dually equivalent to **C*-Alg**.

(9) **Set** is dually equivalent to the category of complete atomic boolean algebras (see Exercise 14H).

## EXERCISES

14A. Show that a category $\mathscr{A}$ is isomorphic with a subcategory of $\mathscr{B}$ if and only if there is an embedding $F: \mathscr{A} \to \mathscr{B}$.

14B. Show that each of the categories **Set**, **Grp**, and **Top** is isomorphic to a subcategory of each of the other two, but that no two of these categories are isomorphic (or even equivalent) to each other.

14C. *Finitely Generated Spaces*
A topological space is said to be **finitely generated** provided that it is a topological quotient of a disjoint topological union of finite spaces.
(a) Prove that for any space $X$, the following are equivalent:
(i) $X$ is finitely generated.
(ii) The intersection of any family of open sets in $X$ is open.
(iii) If $A \subset X$, then

$$A^- = \bigcup_{a \in A} \{a\}^-$$

(where "$^-$" denotes closure).
(b) Prove that the full subcategory of **Top** consisting of all finitely generated spaces (resp. finitely generated $T_0$-spaces) is isomorphic with the category **QOS** of quasi-ordered sets (resp. **POS** of partially-ordered sets).

14D. Prove that an isomorphism $F: \mathscr{A} \to \mathscr{A}$ on a category is not necessarily naturally isomorphic to $1_{\mathscr{A}}$. [See Exercise 13L.]

14E. Prove that cardinality of small categories is not preserved by equivalences; in fact, if $\alpha$ and $\beta$ are any infinite cardinal numbers, then there are equivalent categories $\mathscr{A}$ and $\mathscr{B}$ such that $card\,(Mor(\mathscr{A})) = \alpha$ and $card\,(Mor(\mathscr{B})) = \beta$.

14F. Determine whether or not "smallness" is a "categorical property", i.e., whether or not it is preserved by equivalences.

14G. Show that equivalences between categories preserve initial, terminal, zero, projective, and injective objects, separators and coseparators, monomorphisms, epimorphisms, constant morphisms, zero morphisms, connectedness, and pointedness.

14H. *The Category* $\mathbf{Set}^{op}$

(a) Prove that **Set** is not self-dual.

(b) Prove that $\mathbf{Set}^{op}$ is equivalent to the category of complete, atomic† boolean algebras and complete boolean homomorphisms. [*Hint:* Each complete atomic boolean algebra $A$ is isomorphic to the complete boolean algebra of all subsets of the set of atoms of $A$.]

(c) Prove that $\mathbf{Set}^{op}$ can be embedded in **Set** and that **Set** can be embedded in $\mathbf{Set}^{op}$.

14I. Show that if $X$ is an initial object of a category $\mathscr{C}$, then the comma category $(X, \mathscr{C})$ is isomorphic to $\mathscr{C}$.

14J. *Equivalence of Concrete Categories*

Let $(\mathscr{A}, U)$ and $(\mathscr{B}, V)$ be concrete categories. Discuss the relationships between the following concepts:

(1) There exists an isomorphism $H: \mathscr{A} \to \mathscr{B}$ such that $U = V \circ H$.

(2) There exists an equivalence $H: \mathscr{A} \to \mathscr{B}$ such that $U = V \circ H$.

(3) There exist isomorphisms $H: \mathscr{A} \to \mathscr{B}$ and $K$: **Set** $\to$ **Set** such that $K \circ U = V \circ H$.

(4) There exist equivalences $H: \mathscr{A} \to \mathscr{B}$ and $K$: **Set** $\to$ **Set** such that $K \circ U = V \circ H$.

(5) There exists an equivalence $H: \mathscr{A} \to \mathscr{B}$ and an isomorphism $K$: **Set** $\to$ **Set** such that $K \circ U = V \circ H$.

(6) There exists an isomorphism $H: \mathscr{A} \to \mathscr{B}$ and an equivalence $K$: **Set** $\to$ **Set** such that $K \circ U = V \circ H$.

(7) There exists an isomorphism $H: \mathscr{A} \to \mathscr{B}$ such that $U \cong V \circ H$.

(8) There exists an equivalence $H: \mathscr{A} \to \mathscr{B}$ such that $U \cong V \circ H$.

Concrete categories which satisfy (8) above are called **equivalent concrete categories**.

## §15 FUNCTOR CATEGORIES

### 15.1 DEFINITION

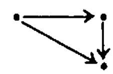
If $\mathscr{A}$ and $\mathscr{B}$ are categories, then the full subquasicategory of $\mathscr{FUNC}$ whose objects are precisely the functors from $\mathscr{A}$ to $\mathscr{B}$ is denoted by $[\mathscr{A}, \mathscr{B}]$ or by $\mathscr{B}^{\mathscr{A}}$ and is called the **(quasi)category of functors from $\mathscr{A}$ to $\mathscr{B}$** or the **functor (quasi)category** $[\mathscr{A}, \mathscr{B}]$.

† A complete boolean algebra is called **atomic** if and only if each of its elements $x$ is the supremum of all of the atoms $a$ with $a \leq x$. ($a$ is called an **atom** provided that it is an immediate successor of 0; i.e., $0 \neq a$ and if $0 < y \leq a$, then $y = a$.)

**15.2 EXAMPLES**

(1) If the category with just one morphism is denoted by **1**, then for any category $\mathscr{C}$, $[\mathbf{1}, \mathscr{C}]$ is isomorphic to $\mathscr{C}$.

(2) If the category $\bullet \longrightarrow \bullet$ is denoted by **2**, then for any category $\mathscr{C}$, the functor category $[\mathbf{2}, \mathscr{C}]$ is isomorphic to the arrow-category $\mathscr{C}^2$ (4.16).

(3) If the category  is denoted by **3**, then for any category $\mathscr{C}$, the functor category $[\mathbf{3}, \mathscr{C}]$ is isomorphic to the triangle-category $\mathscr{C}^3$ (4.17).

Notice that we now have an "internal *Hom*-functor" for $\mathscr{CAT}$, namely, $Hom: \mathscr{CAT}^{op} \times \mathscr{CAT} \to \mathscr{CAT}$, where $Hom(\mathscr{A}, \mathscr{B}) = [\mathscr{A}, \mathscr{B}]$ and $Hom(F, G)(H) = G \circ H \circ F$.

**15.3 PROPOSITION**

*If $\mathscr{A}$ is small, then for any category $\mathscr{B}$, the quasicategory $[\mathscr{A}, \mathscr{B}]$ is isomorphic to a category.*

*Proof:* A functor $F: \mathscr{A} \to \mathscr{B}$ is actually a triple $F = (\mathscr{A}, \bar{F}, \mathscr{B})$ where $\bar{F}$ is a function from $Mor\,\mathscr{A}$ to $Mor\,\mathscr{B}$. Associate with each such functor $F$, the graph $G(\bar{F})$ of $\bar{F}$. Since $Mor\,\mathscr{A}$ is a set, so is $Mor\,\mathscr{A} \times \bar{F}[Mor\,\mathscr{A}]$ (1.1(3) and (4)). Hence, since

$$G(\bar{F}) \subset Mor\,\mathscr{A} \times \bar{F}[Mor\,\mathscr{A}],$$

it must be a set (1.1(1)). Similarly, to each natural transformation $\eta = (F, \bar{\eta}, H)$ between objects of $[\mathscr{A}, \mathscr{B}]$ associate the graph $G(\bar{\eta})$ of $\bar{\eta}$. Since

$$G(\bar{\eta}) \subset Ob\,\mathscr{A} \times \bar{\eta}[Ob\,\mathscr{A}],$$

it must also be a set. Let $\mathscr{C} = (\mathscr{O}, \mathscr{M}, dom, cod, \circ)$, where

$$\mathscr{O} = \{G(F) \mid F: \mathscr{A} \to \mathscr{B}\}$$

$$\mathscr{M} = \{G(\eta) \mid \eta: F \to H \quad \text{where} \quad F, H \in \mathscr{O}\}$$

$dom: \mathscr{M} \to \mathscr{O}$ and $cod: \mathscr{M} \to \mathscr{O}$ are defined by:

$$dom(G(\eta)) = G(dom\,\eta) \qquad \text{and} \qquad cod(G(\eta)) = G(cod\,\eta),$$

and

$$\circ: \{(f, g) \in \mathscr{M} \times \mathscr{M} \mid dom(f) = cod(g)\} \longrightarrow \mathscr{M}$$

is defined by

$$G(\eta) \circ G(\varepsilon) = G(\eta \circ \varepsilon).$$

Since all members of $\mathscr{O}$ and $\mathscr{M}$ are sets, $\mathscr{O}$ and $\mathscr{M}$ must be classes (1.2). It is easily verified that $\mathscr{C}$ satisfies the matching, associativity, and identity existence conditions (3.1). To show that each morphism class is a set, suppose that $F, H: \mathscr{A} \to \mathscr{B}$. Let

$$S = \bigcup\{hom_{\mathscr{B}}(F(A), H(A)) \mid A \in Ob\,\mathscr{A}\}.$$

Since $S$ is a union of a family of sets indexed by a set, it is a set (1.1(4)). Thus, $\mathscr{P}(Ob\,\mathscr{A} \times S)$ is a set (1.1(2) and (3)). But

$$hom_{\mathscr{C}}(G(F), G(H)) = \{G(\eta) \mid \eta\colon F \to H\} \subset \mathscr{P}(Ob\,\mathscr{A} \times S),$$

so that it must be a set (1.1(4)(i)). □

**15.4 PROPOSITION**

*If* $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ *are functors and* $\eta\colon F \to G$ *is a natural transformation, then* $\eta$ *is an isomorphism in* $\mathscr{B}^{\mathscr{A}}$ *if and only if it is a natural isomorphism.*

*Proof:* Immediate from Theorem 13.6. □

Next we will see that in general a morphism $\eta$ in $\mathscr{B}^{\mathscr{A}}$ inherits the categorical properties that are common to all of the $\eta_A$, $A \in Ob(\mathscr{A})$; whereas the category $\mathscr{B}^{\mathscr{A}}$ itself often inherits the categorical properties of $\mathscr{B}$.

**15.5 PROPOSITION**

*A morphism* $\eta$ *in* $[\mathscr{A}, \mathscr{B}]$ *has one of the properties: "isomorphism", "monomorphism", "epimorphism", "bimorphism", "constant morphism", "coconstant morphism", or "zero morphism", if for each* $A \in Ob(\mathscr{A})$, $\eta_A$ *has the corresponding property.*

*Proof:* The proof for isomorphisms is immediate from Proposition 15.4. If each $\eta_A$ is a monomorphism and $\xi$ and $\delta$ are morphisms in $\mathscr{B}^{\mathscr{A}}$ such that $\eta \circ \xi = \eta \circ \delta$, then for each $A \in Ob(\mathscr{A})$, $\eta_A \circ \xi_A = \eta_A \circ \delta_A$. Hence, for each $A$, $\xi_A = \delta_A$; i.e., $\xi = \delta$. The proofs for other cases are left as an exercise. □

**15.6 PROPOSITION**

*If the category* $\mathscr{B}$ *has one of the properties: "has an initial object", "has a terminal object", "has a zero object", or "is pointed", then for any category* $\mathscr{A}$, $[\mathscr{A}, \mathscr{B}]$ *has the corresponding property.*

*Proof:* If $\mathscr{B}$ has an initial object $X$, consider the constant functor $F\colon \mathscr{A} \to \mathscr{B}$ defined by $F(f) = 1_X$ for all $f \in Mor(\mathscr{A})$. Then for any $G \in Ob(\mathscr{B}^{\mathscr{A}})$, $\eta = (\eta_A)\colon F \to G$ defined by: $\eta_A =$ "the unique morphism from $X$ to $G(A)$", is clearly a natural transformation since for each $A \xrightarrow{f} A'$ the square

$$\begin{array}{ccc} F(A) = X & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f) = 1_X}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A') = X & \xrightarrow[\eta_{A'}]{} & G(A') \end{array}$$

must commute. (There is only one morphism from $X$ to $G(A')$.) For the same reason, it is clear that $\eta$ is the only natural transformation from $F$ to $G$. Thus, $\mathscr{B}^{\mathscr{A}}$ has an initial object.

The proof for terminal objects is similar. If $\mathscr{B}$ is pointed, let $C_{BB'}$ denote the unique zero morphism from $B$ to $B'$ (8.8). For any pair of functors

$$F, G: \mathscr{A} \to \mathscr{B},$$

let $\zeta_{FG} = (\zeta_{FG})_A: F \to G$ be defined by $(\zeta_{FG})_A = C_{F(A)G(A)}$. The square

$$\begin{array}{ccc} F(A) & \xrightarrow{C_{F(A)G(A)}} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow[C_{F(A')G(A')}]{} & G(A') \end{array}$$

commutes, by the uniqueness of the zero morphism from $F(A)$ to $G(A')$. Since each $(\xi_{FG})_A$ is a zero morphism, so is $\xi_{FG}$ (15.5). Thus, $\mathscr{B}^{\mathscr{A}}$ is pointed. □

Functor categories naturally give rise to a new type of bifunctor called the **evaluation functor**, defined below.

**15.7 PROPOSITION**

*For any categories $\mathscr{A}$ and $\mathscr{B}$, there is a (bi)functor*

$$E: \mathscr{B}^{\mathscr{A}} \times \mathscr{A} \to \mathscr{B}$$

*defined by:*

$$E(F, A) = F(A),$$

*and for each $\eta: F \to G$ and $f: A \to A'$*

$$E(\eta, f) = G(f) \circ \eta_A = \eta_{A'} \circ F(f),$$

*called the* **evaluation functor for $\mathscr{B}^{\mathscr{A}}$.**

*Proof:* $E$ is well-defined because $G(f) \circ \eta_A = \eta_{A'} \circ F(f)$, since $\eta$ is a natural transformation, i.e., since the square

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow[\eta_{A'}]{} & G(A') \end{array}$$

commutes.

Consider the identity $(1_F, 1_A)$ on $(F, A)$.

$$E(1_F, 1_A) = F(1_A) \circ 1_{F(A)} = 1_{F(A)} \circ 1_{F(A)} = 1_{F(A)} = 1_{E(F,A)}.$$

Thus, $E$ preserves identities.

Suppose $(F, A) \xrightarrow{(\eta,f)} (G, A') \xrightarrow{(\xi,g)} (H, A'')$.

$$E((\xi, g) \circ (\eta, f)) = E(\xi \circ \eta, g \circ f) = H(g \circ f) \circ (\xi \circ \eta)_A$$
$$= H(g) \circ (H(f) \circ \xi_A) \circ \eta_A = H(g) \circ (\xi_{A'} \circ G(f)) \circ \eta_A$$
$$= (H(g) \circ \xi_{A'}) \circ (G(f) \circ \eta_A) = E(\xi, g) \circ E(\eta, f).$$

Hence, $E$ preserves compositions and is therefore a functor. □

**15.8** If $E: \mathscr{B}^{\mathscr{A}} \times \mathscr{A} \to \mathscr{B}$ is the evaluation bifunctor, then for each $A \in Ob(\mathscr{A})$, the left associated functor $E(\_\_, A): \mathscr{B}^{\mathscr{A}} \to \mathscr{B}$ is called the **evaluation functor relative to** $A$. Clearly,

$$E(\_\_, A)(F) = F(A) \qquad \text{for each} \quad F: \mathscr{A} \to \mathscr{B},$$

and

$$E(\_\_, A)(\eta) = \eta_A \qquad \text{for each} \quad \eta \in Mor(\mathscr{B}^{\mathscr{A}}).$$

Note that for each $F: \mathscr{A} \to \mathscr{B}$, the right associated functor $E(F, \_\_): \mathscr{A} \to \mathscr{B}$ is the functor $F$.

Also note that for any two categories $\mathscr{B}$ and $\mathscr{A}$, there is a **"constant functor" functor** $C: \mathscr{B} \to \mathscr{B}^{\mathscr{A}}$ defined by:

*for each $\mathscr{B}$-object, $B$, $C(B)$ is the constant functor from $\mathscr{A}$ to $\mathscr{B}$ whose value at each $\mathscr{A}$-object is $B$ and whose value at each $\mathscr{A}$-morphism is $1_B$,*

*for each $\mathscr{B}$-morphism, $B \xrightarrow{f} \hat{B}$, $C(f)$ is the natural transformation from $C(B)$ to $C(\hat{B})$ defined by:*

$$(C(f))_A = f \qquad \textit{for each} \quad \mathscr{A}\textit{-object } A.$$

**15.9 THEOREM**

*If $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are categories, then the functor (quasi)categories $(\mathscr{C}^{\mathscr{B}})^{\mathscr{A}}$ and $\mathscr{C}^{\mathscr{A} \times \mathscr{B}}$ are isomorphic.*

*Proof:* Define

$$\Gamma: [\mathscr{A}, [\mathscr{B}, \mathscr{C}]] \to [\mathscr{A} \times \mathscr{B}, \mathscr{C}]$$

by

$$(\Gamma(F))(A, B) = (F(A))(B)$$

and if

$$(\alpha, \beta): (A, B) \to (A', B')$$

$$(\Gamma(F))(\alpha, \beta) = F(A')(\beta) \circ F(\alpha)_B = F(\alpha)_{B'} \circ F(A)(\beta).$$

And if $\eta: F \to G$

$$\Gamma(\eta)(A, B) = (\eta_A)_B.$$

By straightforward arguments, it can be shown that $\Gamma(F)$ is a functor from $\mathscr{A} \times \mathscr{B}$ to $\mathscr{C}$ and that $\Gamma(\eta)$ is a natural transformation from $\Gamma(F)$ to $\Gamma(G)$. Clearly,

$$\Gamma(1_F)(A, B) = ((1_F)_A)_B = (1_{F(A)})_B = 1_{F(A)(B)},$$

so that $\Gamma$ preserves identities. If $F \xrightarrow{\eta} G \xrightarrow{\xi} A$, then by the definition of composition of natural transformations (13.3), we have that for each

$$(A, B) \in Ob(\mathscr{A} \times \mathscr{B}),$$

$$\begin{aligned}\Gamma(\xi \circ \eta)(A, B) &= ((\xi \circ \eta)_A)_B = (\xi_A \circ \eta_A)_B = (\xi_A)_B \circ (\eta_A)_B \\ &= \Gamma(\xi)(A, B) \circ \Gamma(\eta)(A, B) = (\Gamma(\xi) \circ \Gamma(\eta))(A, B).\end{aligned}$$

Hence, $\Gamma(\xi \circ \eta) = \Gamma(\xi) \circ \Gamma(\eta)$, so that $\Gamma$ preserves compositions and is thus a functor.

Now define

$$\Delta : [\mathscr{A} \times \mathscr{B}, \mathscr{C}] \to [\mathscr{A}, [\mathscr{B}, \mathscr{C}]]$$

by

$$\Delta(H)(A) = \text{the right associated functor } H(A, \_\_): \mathscr{B} \to \mathscr{C}$$

and if $\alpha: A \to A'$

$$(\Delta(H)(\alpha))_B = H(\alpha, 1_B)$$

and if $\sigma: H \to K$

$$(\Delta(\sigma)_A)_B = \sigma_{(A,B)}.$$

Again, straightforward arguments can be used to show that $\Delta(H)$ is a functor from $\mathscr{A}$ to $\mathscr{C}^{\mathscr{B}}$ and that $\Delta(\sigma)$ is a natural transformation from $\Delta(H)$ to $\Delta(K)$. Now

$$(\Delta(1_H)_A)_B = (1_H)_{(A,B)} = 1_{H(A,B)} = 1_{(\Delta H(A))(B)}.$$

Thus, $\Delta$ preserves identities. If $H \xrightarrow{\sigma} K \xrightarrow{\tau} L$, then

$$(\Delta(\tau \circ \sigma)_A)_B = (\tau \circ \sigma)_{(A,B)} = \tau_{(A,B)} \circ \sigma_{(A,B)} = ((\Delta\tau \circ \Delta\sigma)_A)_B.$$

Hence, $\Delta$ preserves compositions, and so is a functor. If $\eta$ is a morphism in $[\mathscr{A}, [\mathscr{B}, \mathscr{C}]]$, then

$$((\Delta \circ \Gamma(\eta))_A)_B = (\Delta(\Gamma(\eta))_A)_B = \Gamma(\eta)_{(A,B)} = (\eta_A)_B.$$

Hence,

$$\Delta \circ \Gamma(\eta) = \eta,$$

so that

$$\Delta \circ \Gamma = 1_{[\mathscr{A}, [\mathscr{B}, \mathscr{C}]]}.$$

Likewise, if $\sigma$ is a morphism in $[\mathscr{A} \times \mathscr{B}, \mathscr{C}]$, then

$$(\Gamma \circ \Delta(\sigma))(A, B) = \Gamma(\Delta(\sigma))(A, B) = (\Delta(\sigma)_A)_B = \sigma_{(A,B)}.$$

Hence,

$$\Gamma \circ \Delta(\sigma) = \sigma,$$

so that

$$\Gamma \circ \Delta = 1_{[\mathscr{A} \times \mathscr{B}, \mathscr{C}]}.$$

Consequently, $\Gamma$ is an isomorphism. $\square$

## EXERCISES

15A. Prove that for any categories $\mathscr{A}$ and $\mathscr{B}$, $[\mathscr{A}, \mathscr{B}]$ and $[\mathscr{A}^{op}, \mathscr{B}^{op}]$ are dually equivalent.

15B. In the proof of Theorem 15.9, show that:
(a) For each functor $F: \mathscr{A} \to \mathscr{C}^{\mathscr{B}}$, $\Gamma(F)$ is a functor from $\mathscr{A} \times \mathscr{B}$ to $\mathscr{C}$.

(b) For each morphism $F \xrightarrow{\eta} G$ in $[\mathscr{A}, [\mathscr{B}, \mathscr{C}]]$, $\Gamma(\eta)$ is a natural transformation from $\Gamma(F)$ to $\Gamma(G)$.
(c) For each functor $G: \mathscr{A} \times \mathscr{B} \to \mathscr{C}$, $\Delta(G)$ is a functor from $\mathscr{A}$ to $\mathscr{C}^{\mathscr{B}}$.

(d) For each morphism $H \xrightarrow{\sigma} K$ in $[\mathscr{A} \times \mathscr{B}, \mathscr{C}]$, $\Delta(\sigma)$ is a natural transformation from $\Delta(H)$ to $\Delta(K)$.

15C. Prove that for any two non-empty categories, $\mathscr{A}$ and $\mathscr{B}$, the "constant functor" functor $C: \mathscr{B} \to \mathscr{B}^{\mathscr{A}}$ is a section in $\mathscr{CAT}$, and for each $\mathscr{A}$-object $A$, the evaluation functor relative to $A$, $E(\_\_, A): \mathscr{B}^{\mathscr{A}} \to \mathscr{B}$, is a retraction in $\mathscr{CAT}$.