IV

Special Morphisms and Special Objects

Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial.

——P. Freyd†

In the category of sets, certain special types of functions play distinguished roles; these are:

identity functions
injective functions
surjective functions
bijective functions
constant functions

In Chapter III, we encountered the identity morphisms, which in arbitrary categories are the obvious analogue of the identity functions in Set. For the other classes of functions, which are usually defined in terms of elements of their domains and codomains, we will now furnish "element-free" characterizations and will investigate the corresponding categorical concepts. Likewise, we will find suitable categorical analogues for some distinguished objects in **Set**; namely, the empty set and the singleton sets.

§5 SECTIONS, RETRACTIONS, AND ISOMORPHISMS

Sections

5.1 MOTIVATING PROPOSITION

If $f: A \longrightarrow B$ is a function from a non-empty set A to the set B, then the following are equivalent:

† From *Proceedings of the Conference on Categorical Algebra.*

(1) *f is injective (i.e., one-to-one).*

(2) *There exists some function $g: B \to A$ such that $g \circ f = 1_A$ (i.e., f has a "left inverse" with respect to function composition).* □

5.2 DEFINITION

A morphism $A \xrightarrow{f} B$ in a category $\mathscr{C}$ is said to be a **section in $\mathscr{C}$** (or a **$\mathscr{C}$-section**) provided that there exists some $\mathscr{C}$-morphism $B \xrightarrow{g} A$ such that $g \circ f = 1_A$ (i.e., f has a "left inverse" in $\mathscr{C}$).

5.3 EXAMPLES

(1) A morphism in **Set** is a section if and only if it is injective and is not the empty function from the empty set to a non-empty set.

(2) A morphism $A \xrightarrow{f} B$ in R-**Mod** is a section if and only if it is injective and $f[A]$ is a direct summand of B. In other words, the sections in R-**Mod** are (up to isomorphism) the embeddings of direct summands.

(3) A morphism $X \xrightarrow{f} Y$ in **Top** is a section if and only if f is a topological embedding and $f[X]$ is a retract of Y.† In other words, the sections in **Top** are (up to homeomorphism) the embeddings of retracts.

(4) If X and Y are sets (resp. topological spaces) and if $a \in Y$ then the function $f: X \to X \times Y$ defined by $f(x) = (x, a)$ is a section in **Set** (resp. **Top**). (What is its "natural" left inverse?) [Note that the image of f is a "slice" or "cross-section" of the product, which is in one-to-one correspondence (resp. homeomorphic) to X. This motivates our use of the word "section" in Definition 5.2.]

(5) The sections described in (4) are just special cases of the following situation: Let $f: X \to Y$ be a morphism in **Set** (resp. **Top**, **Grp**, R-**Mod**). Consider the graph of f as a subset (resp. subspace, subgroup, submodule) of the product $X \times Y$. Then the embedding of X into $X \times Y$ defined by $x \mapsto (x, f(x))$ is a section in the category in question.

5.4 PROPOSITION

If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are sections in a category $\mathscr{C}$, then $A \xrightarrow{g \circ f} C$ is a section.

Proof: Let $B \xrightarrow{h} A$ and $C \xrightarrow{k} B$ be morphisms such that $h \circ f = 1_A$ and $k \circ g = 1_B$. Then

$$(h \circ k) \circ (g \circ f) = h \circ (k \circ g) \circ f = h \circ 1_B \circ f = h \circ f = 1_A.$$

Thus $(g \circ f)$ has a left inverse. □

5.5 PROPOSITION

If f and g are morphisms of a category $\mathscr{C}$ and $g \circ f$ is a section, then f is a section.

† A subspace A of a topological space Y is called a **retract** of Y if and only if there is some continuous function $r: Y \to A$ that leaves each point of A fixed. Such a function, r, is called a **topological retraction**.

Proof: By the definition of section we know that there is a morphism h such that $h \circ (g \circ f)$ is an identity. Thus, by the associativity of composition, $(h \circ g) \circ f$ is an identity; hence, f is a section. □

Retractions

5.6 MOTIVATING PROPOSITION

If $f: A \to B$ is a function, then the following are equivalent:

(1) *f is surjective.*

(2) *There is some function $g: B \to A$ such that $f \circ g = 1_B$ (i.e., f has a "right inverse" with respect to function composition).*

Proof: By the Axiom of Choice there is a function that assigns to each $b \in B$ some member of the set $f^{-1}[\{b\}]$. □

5.7 DEFINITION

A $\mathscr{C}$-morphism $A \xrightarrow{f} B$ is said to be a **retraction in $\mathscr{C}$** (or a **$\mathscr{C}$-retraction**) provided that there exists some $\mathscr{C}$-morphism $B \xrightarrow{g} A$ such that $f \circ g = 1_B$ (i.e., f has a "right inverse" in $\mathscr{C}$).

5.8 EXAMPLES

(1) A morphism in **Set** is a retraction if and only if it is surjective.

(2) A morphism $f: A \to B$ in R-**Mod** is a retraction if and only if there exists a projection† p of A onto a submodule S of A and an isomorphism $h: S \to B$ such that $f = h \circ p$. In other words, the retractions in R-**Mod** are (up to isomorphism) exactly the projections of modules onto their direct summands.††

(3) A morphism f in **Top** is a retraction if and only if there is a continuous retraction r and a homeomorphism h such that $f = h \circ r$. In other words, the retractions in **Top** are (up to homeomorphism) exactly the topological retractions. [This motivates our use of the word "retraction" in Definition 5.7.]

5.9 PROPOSITION

Section and retraction are dual notions.

Proof: Let $S(\mathscr{C})$ be the statement:

$f \in Mor(\mathscr{C})$ and there exists some $g \in Mor(\mathscr{C})$ such that $g \circ_{\mathscr{C}} f$ is a $\mathscr{C}$-identity.

Then $S(\mathscr{C}^{op})$ is the statement:

$f \in Mor(\mathscr{C}^{op})$ and there exists some $g \in Mor(\mathscr{C}^{op})$ such that $g \circ_{\mathscr{C}^{op}} f$ is a $\mathscr{C}^{op}$-identity.

Translating this into a statement about $\mathscr{C}$, we obtain:

$f \in Mor(\mathscr{C})$ and there exists some $g \in Mor(\mathscr{C})$ such that $f \circ_{\mathscr{C}} g$ is a $\mathscr{C}$-identity.

This is precisely the statement that f is a retraction in $\mathscr{C}$. □

† If S is a submodule of A, then a projection of A onto S is a surjective homomorphism $p: A \to S$ that leaves each point of S fixed. Such a homomorphism exists if and only if S is a direct summand of A.

†† Thus, an R-module B is a "retract" of A if and only if it is a direct summand of A. Also we have seen that B is a "sect" of A if and only if it is a direct summand of A (5.3(2)). Consequently, in R-Mod an object B is a "retract" of the object A if and only if it is a "sect" of A. A corresponding statement is true for every category. (Why?)

5.10 PROPOSITION

If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are $\mathscr{C}$-retractions, then $A \xrightarrow{g\circ f} C$ is a $\mathscr{C}$-retraction.

Proof: If f and g are $\mathscr{C}$-retractions, then according to Proposition 5.9 they are $\mathscr{C}^{op}$-sections and thus their composition $f \circ_{\mathscr{C}^{op}} g$ in $\mathscr{C}^{op}$ is a $\mathscr{C}^{op}$-section (5.4); i.e., $g \circ_{\mathscr{C}} f = f \circ_{\mathscr{C}^{op}} g$ is a $\mathscr{C}$-retraction. □

From now on we will not always indicate the proofs of dual propositions or theorems, since they are all an application of the duality principle (4.15). After a while we will sometimes not even provide the dual statements for theorems, leaving this task as an implied exercise for the reader.

5.11 PROPOSITION

If f and g are $\mathscr{C}$-morphisms and $g \circ f$ is a retraction, then g is a retraction.

Proof: Dualize Proposition 5.5. □

Isomorphisms

5.12 MOTIVATING PROPOSITION

If $f: A \to B$ is a function, then the following are equivalent:

(1) *f is bijective.*

(2) *There exists some function $g: B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.* □

5.13 DEFINITION

A $\mathscr{C}$-morphism is said to be an **isomorphism in $\mathscr{C}$** (or a **$\mathscr{C}$-isomorphism**) provided that it is both a $\mathscr{C}$-section and a $\mathscr{C}$-retraction (i.e., it has both a "left inverse" and a "right inverse" in $\mathscr{C}$).

5.14 EXAMPLES

(1) In any category, every identity is an isomorphism.

(2) A morphism in **Set** is an isomorphism if and only if it is bijective.

(3) A morphism in **Grp** is an isomorphism if and only if it is a group-theoretic isomorphism.

(4) A morphism in **Top** is an isomorphism if and only if it is a homeomorphism

(5) A morphism in $\mathbf{BanSp}_1$ is an isomorphism if and only if it is a homeomorphic linear isomorphism, and a morphism in $\mathbf{BanSp}_2$ is an isomorphism if and only if it is an isometric linear isomorphism.

(6) A monoid is a group if and only if, considered as a category (3.5(7)), each of its morphisms is an isomorphism.

5.15 PROPOSITION

Isomorphism is a self-dual notion.

Proof: The notions of section and retraction are duals of each other (5.9). □

5.16 PROPOSITION

In any category, the composition of isomorphisms is an isomorphism.

Proof: Immediate from the fact that sections and retractions are closed under composition (5.4 and 5.10). □

5.17 PROPOSITION

If f is a $\mathscr{C}$-morphism, then the following are equivalent:

(1) *f is a $\mathscr{C}$-isomorphism.*

(2) *f has exactly one right inverse, h, and exactly one left inverse, k, and $h = k$.*

Proof: Clearly (2) implies (1). To show that (1) implies (2), we know by definition that f has some right inverse h and some left inverse k. We need only show that $h = k$. Clearly

$$k = k \circ 1 = k \circ (f \circ h) = (k \circ f) \circ h = 1 \circ h = h. \quad \square$$

Because of the above proposition, we may speak of the inverse of an isomorphism f. It is usually denoted by f^{-1}.

5.18 PROPOSITION

If f is an isomorphism, then f^{-1} is an isomorphism and $f = (f^{-1})^{-1}$. □

5.19 DEFINITION

An object A of a category $\mathscr{C}$ is said to be **$\mathscr{C}$-isomorphic** with an object B of $\mathscr{C}$ provided that there exists some $\mathscr{C}$-isomorphism $f: A \to B$.

5.20 PROPOSITION

For any category $\mathscr{C}$, "is isomorphic with" yields an equivalence relation on $Ob(\mathscr{C})$.

Proof: Reflexivity holds since identities are isomorphisms. Symmetry follows from the fact that if f is an isomorphism, then f^{-1} is one also, and transitivity holds since isomorphisms are closed under composition. □

5.21 DEFINITION

Let $\mathscr{B}$ be a subcategory of $\mathscr{C}$.

(1) $\mathscr{B}$ is said to be a **dense subcategory** of $\mathscr{C}$ provided that for each $\mathscr{C}$-object C, there is some $\mathscr{B}$-object B such that B is $\mathscr{C}$-isomorphic with C.

(2) $\mathscr{B}$ is said to be an **isomorphism-closed subcategory** of $\mathscr{C}$ provided that every $\mathscr{C}$-object that is isomorphic with some $\mathscr{B}$-object is itself a $\mathscr{B}$-object.

5.22 EXAMPLES

(1) The category of all cardinal numbers and functions between them is a dense subcategory of **Set**.

(2) The category of all subgroups of permutation groups and homomorphisms between them is a dense subcategory of **Grp**.

(3) If F is a field, then the full subcategory of all finite powers F^n of F is a dense subcategory of the category of all finite dimensional vector spaces over F.

(4) If F is a field, then the full subcategory of all powers F^I of F is a dense subcategory of F-**Mod**.

(5) If X is a topological space with three points and exactly three open sets, then the full subcategory of all subspaces of powers X^I of X is a dense subcategory of **Top**.

(6) $\mathbf{BanSp}_2$ is both a dense and an isomorphism-closed subcategory of $\mathbf{BanSp}_1$.

(7) A full subcategory $\mathscr{B}$ of a category $\mathscr{C}$ is both dense and isomorphism-closed in $\mathscr{C}$ if and only if $\mathscr{B} = \mathscr{C}$.

EXERCISES

5A. Show that if f and g are isomorphisms in a category $\mathscr{C}$, then $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ (i.e., when either side is defined, then so is the other and they are equal).

5B. Show that in general a section may have several left inverses and a retraction may have several right inverses.

5C. Let f be a morphism in the category of all sets which have at least two elements. Show that the following are equivalent:

(a) f is an isomorphism.
(b) f has exactly one right inverse.
(c) f has exactly one left inverse.

Do these same equivalences hold in the category of all topological spaces with at least two members?

5D. Let f and g be $\mathscr{C}$-morphisms. Show that if $g \circ f$ is an isomorphism, then f is a section and g is a retraction, but not conversely.

5E. Show that if the monoid of natural numbers under addition is considered as a category (3.5(7)), then zero is the only section, the only retraction, and thus the only isomorphism.

5F. Let $\mathscr{B}$ be a subcategory of $\mathscr{C}$.

(a) Prove that any $\mathscr{B}$-section (resp. $\mathscr{B}$-retraction, $\mathscr{B}$-isomorphism) is a $\mathscr{C}$-section (resp. $\mathscr{C}$-retraction, $\mathscr{C}$-isomorphism).
(b) If $\mathscr{B}$ is a full subcategory of $\mathscr{C}$, show that every $\mathscr{B}$-morphism that is a $\mathscr{C}$-section (resp. $\mathscr{C}$-retraction, $\mathscr{C}$-isomorphism) is necessarily a $\mathscr{B}$-section (resp. $\mathscr{B}$-retraction, $\mathscr{B}$-isomorphism).
(c) Show that without the requirement that $\mathscr{B}$ is full, (b) above is false.

5G. Prove that if $\bar{\mathscr{C}}$ is a quotient category for $\mathscr{C}$ and if f is a $\mathscr{C}$-section (resp. $\mathscr{C}$-retraction, $\mathscr{C}$-isomorphism), then the equivalence class $\bar{f}$ is a $\bar{\mathscr{C}}$-section (resp. $\bar{\mathscr{C}}$-retraction, $\bar{\mathscr{C}}$-isomorphism).

5H. A $\mathscr{C}$-morphism g is said to be a **quasi-inverse** for the $\mathscr{C}$-morphism f if and only if $f \circ g \circ f = f$. Prove that every $\mathscr{C}$-morphism that has a quasi-inverse is itself the quasi-inverse of some $\mathscr{C}$-morphism.

§6 MONOMORPHISMS, EPIMORPHISMS, AND BIMORPHISMS

Monomorphisms

6.1 MOTIVATING PROPOSITION

If $f: A \to B$ is a function on sets, then the following are equivalent:

(1) *f is injective.*

(2) *For all functions h and k such that $f \circ h = f \circ k$, it follows that $h = k$ (i.e., f is "left-cancellable" with respect to function composition).*

Proof: Clearly (1) implies (2). If f satisfies (2) and $a, b \in A$ such that $f(a) = f(b)$, consider functions from a singleton set into A, one of which has image $\{a\}$ and the other of which has image $\{b\}$. □

6.2 DEFINITION

A $\mathscr{C}$-morphism $A \xrightarrow{f} B$ is said to be a **monomorphism in $\mathscr{C}$** (or a **$\mathscr{C}$-monomorphism**) provided that for all $\mathscr{C}$-morphisms h and k such that $f \circ h = f \circ k$, it follows that $h = k$ (i.e., f is "left-cancellable" with respect to composition in $\mathscr{C}$).

6.3 EXAMPLES

(1) Every morphism in a concrete category that is an injective function on the underlying sets is a monomorphism.

(2) In **Set**, **Grp**, **SGrp**, **Ab**, R-**Mod**, **Rng**, **POS**, **Top**, $\mathbf{Top_2}$, $\mathbf{CompT_2}$, **LinTop**, $\mathbf{BanSp_1}$, and $\mathbf{BanSp_2}$, the monomorphisms are precisely the morphisms which are injective on the underlying sets. Notice that in **Set**, **Grp**, **SGrp**, **Ab**, R-**Mod**, **Rng**, $\mathbf{CompT_2}$, and $\mathbf{BanSp_1}$, the monomorphisms are "essentially" the embeddings of substructures, but in **POS**, **Top**, $\mathbf{Top_2}$, **LinTop**, and $\mathbf{BanSp_2}$, there are monomorphisms which are not embeddings. A satisfactory categorical concept for "embeddings" will be discussed later (see 34G).

(3) In the category $\mathscr{A}$ of divisible abelian groups and group homomorphisms, there are monomorphisms which are not injective on the underlying sets. [Consider the natural quotient $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$, where $\mathbf{Q}$ (= the rational numbers) and $\mathbf{Z}$ (= the integers) are each considered as abelian groups under addition.]

(4) In the category $\mathscr{C}$ of pointed connected spaces and continuous base-point-preserving functions, there are monomorphisms that are not injective on the underlying sets. [Consider the pointed space $(\mathbf{R}, 0)$ of the real numbers with base point 0 and the pointed space $(X, 1)$ where X is the circle S^1 represented as the space of all complex numbers with modulus 1. Then $x \mapsto e^{ix}$ defines a morphism $p: (\mathbf{R}, 0) \to (X, 1)$ that is a $\mathscr{C}$-monomorphism but not an injective function. Notice that p is a covering projection and the "unique lifting property" of covering projections† is equivalent to the statement that each covering projection in $\mathscr{C}$ is a $\mathscr{C}$-monomorphism.]

† See E. H. Spanier, *Algebraic Topology*, New York: McGraw-Hill, 1966, p. 67.

(5) There is a monomorphism $X \xrightarrow{f} Y$ in **Top** such that the homotopy class $X \xrightarrow{\bar{f}} Y$ of f is not a monomorphism in **hTop** (the homotopy category of topological spaces). [Consider the usual embedding of the bounding circle of a disc into the disc.]

(6) In the category **Field**,† and in any category which is a quasi-ordered class (in the sense of 3.5(6)), every morphism is a monomorphism.

6.4 PROPOSITION

If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are $\mathscr{C}$-monomorphisms, then $A \xrightarrow{g \circ f} C$ is a $\mathscr{C}$-monomorphism.

Proof: If $(g \circ f) \circ h = (g \circ f) \circ k$, then $g \circ (f \circ h) = g \circ (f \circ k)$. Since g is a monomorphism $f \circ h = f \circ k$, and since f is a monomorphism $h = k$. □

6.5 PROPOSITION

If f and g are $\mathscr{C}$-morphisms and $g \circ f$ is a monomorphism, then f is a monomorphism.

Proof: $f \circ h = f \circ k \Rightarrow g \circ (f \circ h) = g \circ (f \circ k) \Rightarrow (g \circ f) \circ h = (g \circ f) \circ k \Rightarrow h = k$. □

6.6 PROPOSITION

Every $\mathscr{C}$-section is also a $\mathscr{C}$-monomorphism.

Proof: If g is a left inverse for f, then

$$\begin{aligned} f \circ h = f \circ k &\Rightarrow g \circ (f \circ h) = g \circ (f \circ k) \\ &\Rightarrow (g \circ f) \circ h = (g \circ f) \circ k \\ &\Rightarrow 1 \circ h = 1 \circ k \Rightarrow h = k. \quad \square \end{aligned}$$

The converse of the above proposition does not hold since, for example, in **Top**, the embedding of an open interval into a closed interval is a monomorphism but not a section.

6.7 PROPOSITION

In any category, the following are equivalent:

(1) *f is an isomorphism.*

(2) *f is a monomorphism and a retraction.*

Proof: An isomorphism is a section and a retraction, so that by Proposition 6.6 it is both a monomorphism and retraction. Thus, (1) implies (2). Let f be a monomorphism and a retraction and let g be a right inverse of f.

$$f \circ g = 1 \Rightarrow (f \circ g) \circ f = 1 \circ f = f \circ 1 \Rightarrow f \circ (g \circ f) = f \circ 1$$

so that since f is a monomorphism, $g \circ f = 1$. Hence, f is a section, and therefore an isomorphism. Thus, (2) implies (1). □

† Recall that in each field $0 \neq 1$.

Epimorphisms

6.8 MOTIVATING PROPOSITION

If $f: A \to B$ is a function on sets, then the following are equivalent:

(1) *f is surjective.*

(2) *For all functions h and k such that $h \circ f = k \circ f$, it follows that $h = k$ (i.e., f is "right-cancellable" with respect to function composition).*

Proof: Clearly (1) implies (2). If $f: A \to B$ is not surjective, define functions $h, k: B \to \{1, 2\}$ by:

$$h[B] = \{1\}, \qquad k[f[A]] = \{1\},$$

and

$$k[B - f[A]] = \{2\}.$$

Then, $h \circ f = k \circ f$, but $h \neq k$. □

6.9 DEFINITION

A $\mathscr{C}$-morphism $A \xrightarrow{f} B$ is said to be an **epimorphism in $\mathscr{C}$** (or a **$\mathscr{C}$-epimorphism**) provided that for all $\mathscr{C}$-morphisms h and k such that $h \circ f = k \circ f$, it follows that $h = k$ (i.e., f is "right-cancellable" with respect to the composition in $\mathscr{C}$).

6.10 EXAMPLES

(1) Every morphism in a concrete category which is a surjective function on the underlying sets is an epimorphism.

(2) In **Set**, **Grp**, **Ab**, R-**Mod**, **POS**, **Top**, and **CompT$_2$**, the epimorphisms are precisely the morphisms which are surjective on the underlying sets. [The proof for **Grp** is not immediate (see Exercise 6H); if $A \xrightarrow{f} B$ is an epimorphism in **Ab** or R-**Mod**, let $h, k: B \to B/f[A]$ be the induced quotient map and the zero map, respectively.]

(3) There is an epimorphism $X \xrightarrow{f} Y$ in **Top** such that the homotopy class $X \xrightarrow{\bar{f}} Y$ of f is *not* an epimorphism in **hTop**. [Consider the covering projection of the real line onto the circle, defined by: $x \mapsto e^{ix}$.]

(4) In **Top$_2$** the epimorphisms are precisely the continuous functions with dense images, i.e., the continuous functions $f: A \to B$ for which the closure of $f[A]$ equals B. [If $A \xrightarrow{f} B$ is an epimorphism, let C be the disjoint topological union of two "copies" of B where the corresponding points of the closure of $f[A]$ have been identified, and let h and k be the two natural maps from B to C.] Likewise, in **BanSp$_1$** and **BanSp$_2$** the epimorphisms are precisely the morphisms with dense images.

(5) In the category of torsion-free abelian groups, a morphism $A \xrightarrow{f} B$ is an epimorphism if and only if the factor group $B/f[A]$ is a torsion group. Thus, in this category, epimorphisms need not be surjective.

(6) In **Rng** and **Sgp** there are epimorphisms that are not surjective; e.g., if **Z** is the integers and **Q** the rationals, then the usual embedding $\mathbf{Z} \xrightarrow{f} \mathbf{Q}$ is an epimorphism in **Rng** and in **SGrp**. [If h and k are homomorphisms such that $h \circ f = k \circ f$ and if $n/m \in \mathbf{Q}$, then

$$\begin{aligned} h(n/m) &= h(n)\cdot h(1/m)\cdot h(1) = k(n)\cdot h(1/m)\cdot k(1) \\ &= k(n)\cdot h(1/m)\cdot k(m)\cdot k(1/m) = k(n)\cdot h(1/m)\cdot h(m)\cdot k(1/m) \\ &= k(n)\cdot h(1)\cdot k(1/m) = k(n)\cdot k(1)\cdot k(1/m) = k(n/m).] \end{aligned}$$

(7) In the category of finite semigroups, there are epimorphisms that are not surjective. [Consider the semigroups $A = \{0, a_{11}, a_{12}, a_{21}, a_{22}\}$ and $B = A - \{a_{12}\}$, each with binary operation $\cdot$ defined by:

$$a_{pq}\cdot a_{mn} = \begin{cases} 0 & \text{if } q \neq m \\ a_{pn} & \text{if } q = m \end{cases}$$

With this operation, A and B are finite semigroups and the inclusion $B \to A$ is an epimorphism (Howie and Isbell, 1967).]

6.11 PROPOSITION

Monomorphism and epimorphism are dual notions.

Proof: Let $S(\mathscr{C})$ be the statement:

$f \in Mor(\mathscr{C})$, *and for all* $h, k \in Mor(\mathscr{C})$, $f \circ h = f \circ k \Rightarrow h = k$.

Then $S(\mathscr{C}^{op})$ interpreted as a statement about $\mathscr{C}$ is:

$f \in Mor(\mathscr{C})$, *and for all* $h, k \in Mor(\mathscr{C})$, $h \circ f = k \circ f \Rightarrow h = k$. □

6.12 PROPOSITION

The composition of $\mathscr{C}$-epimorphisms is a $\mathscr{C}$-epimorphism.

Proof: Dualize Proposition 6.4. □

6.13 PROPOSITION

If $g \circ f$ is a $\mathscr{C}$-epimorphism, then g is a $\mathscr{C}$-epimorphism.

Proof: Dualize Proposition 6.5. □

6.14 PROPOSITION

Every $\mathscr{C}$-retraction is a $\mathscr{C}$-epimorphism.

Proof: Dualize Proposition 6.6. □

6.15 PROPOSITION

In any category, the following are equivalent:

(1) *f is an isomorphism.*

(2) *f is an epimorphism and a section.*

Proof: Dualize Proposition 6.7. □

Even though the notions of epimorphism and monomorphism are dual and can thus always be handled symmetrically, in well-known categories their behavior often appears to be far from symmetric. For instance in most of the concrete categories that we have considered, the monomorphisms are precisely those morphisms that are monomorphisms (i.e., injective functions) on the underlying sets. However, it is quite usual for epimorphisms in concrete categories not to be epimorphisms (i.e., surjective functions) on the underlying sets (e.g., in **SGrp**, **Mon**, **Rng**, $\mathbf{Top_2}$, $\mathbf{BanSp_1}$, and $\mathbf{BanSp_2}$). Actually, there is a good reason for this, which will be explained later (see §30 and Proposition 24.5).

Bimorphisms

6.16 DEFINITION

A $\mathscr{C}$-morphism is said to be a **bimorphism in** $\mathscr{C}$ (or a $\mathscr{C}$-**bimorphism**) provided that it is both a monomorphism and an epimorphism.

6.17 EXAMPLES

(1) For every category $\mathscr{C}$, each $\mathscr{C}$-isomorphism is a $\mathscr{C}$-bimorphism.

(2) For the categories **Set**, **Grp**, **Ab**, R-**Mod**, **POS**, and **Top**, the bimorphisms are precisely those morphisms that are bijective on the underlying sets. Note that in **Top** and **POS** they need not be isomorphisms.

(3) In each quasi-ordered class considered as a category (3.5(6)), every morphism is a bimorphism.

(4) A monoid is cancellative if and only if, considered as a category (3.5(7)), each of its morphisms is a bimorphism.

6.18 DEFINITION

A category is said to be **balanced** provided that each of its bimorphisms is an isomorphism.

6.19 EXAMPLES

(1) **Set**, **Grp**, **Ab**, R-**Mod**, and $\mathbf{CompT_2}$ are balanced.

(2) **Rng**, **Sgp**, $\mathbf{Top_2}$, **Top**, **LinTop**, and **POS** are not balanced. (For the first three, epimorphisms need not be surjective functions; for the last four, monomorphisms need not be embeddings.)

(3) A partially ordered class considered as a category is balanced if and only if it is discrete.

6.20 PROPOSITION

The composition of $\mathscr{C}$*-bimorphisms is a* $\mathscr{C}$*-bimorphism.*

Proof: Monomorphisms and epimorphisms are closed under composition (6.4 and 6.12). □

6.21 PROPOSITION

If $g \circ f$ *is a* $\mathscr{C}$*-bimorphism, then* f *is a monomorphism and* g *is an epimorphism, but not conversely.* □

Subobjects and Quotient Objects

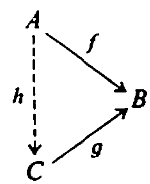
6.22 DEFINITION

A **subobject** of an object $B \in Ob(\mathscr{C})$ is a pair (A, f) where $A \xrightarrow{f} B$ is a monomorphism. If f also happens to be a section, then (A, f) is sometimes called a **sect** of B.

DUAL NOTION: **quotient object**; **retract**. [I.e., (f, A)† is a **quotient object** of B provided that $B \xrightarrow{f} A$ is an epimorphism, and (f, A) is a **retract** of B provided that f is a retraction.]

6.23 DEFINITION

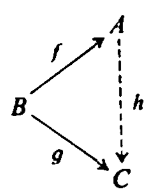
(1) If (A, f) and (C, g) are subobjects of B, then (A, f) is said to be **smaller than** (C, g)—denoted by $(A, f) \leq (C, g)$—if and only if there exists some morphism $A \xrightarrow{h} C$ such that the triangle



commutes.

(2) If $(A, f) \leq (C, g)$ and $(C, g) \leq (A, f)$, then (A, f) and (C, g) are said to be **isomorphic subobjects** of B; denoted by $(A, f) \approx (C, g)$.

DUALLY: (1)* The quotient object (f, A) is **larger than** the quotient object (g, C)—denoted by $(f, A) \geq (g, C)$—if and only if there exists some morphism $A \xrightarrow{h} C$ such that the triangle



commutes.

(2)* (f, A) and (g, C) are **isomorphic quotient objects**—denoted by $(f, A) \approx (g, C)$—if and only if $(f, A) \geq (g, C)$ and $(g, C) \geq (f, A)$.

Notice that even though one cannot formally take a subobject of a subobject (since a subobject is a pair rather than an object), it is clear that if (B, f) is a subobject of A and (C, g) is a subobject of B, then $(C, f \circ g)$ is a subobject of A. Hence, in this sense a subobject of a subobject is a subobject.

† For quotient objects, we write the pair as (f, A) rather than (A, f) as a mnemonic device to aid in recalling that A is the codomain of f rather than the domain of f.

6.24 PROPOSITION

Subobjects (A, f) and (C, g) of B are isomorphic subobjects of B if and only if there exists a unique isomorphism $A \xrightarrow{h} C$ such that $g \circ h = f$.

Proof: Suppose that (A, f) and (C, g) are isomorphic subobjects. Since $(A, f) \leq (C, g)$, there is a morphism h such that $g \circ h = f$. Since f is a monomorphism, h must be also (6.5). $(C, g) \leq (A, f)$ implies that there is a morphism k such that $f \circ k = g$. Now

$$g \circ (h \circ k) = (g \circ h) \circ k = f \circ k = g = g \circ 1_C.$$

Thus since g is a monomorphism, $h \circ k = 1_C$. Hence, h is a retraction and a monomorphism, so it is an isomorphism (6.7). Uniqueness of h follows from the fact that g is a monomorphism. Conversely, if $A \xrightarrow{h} C$ is an isomorphism such that $g \circ h = f$, then clearly $(A, f) \leq (C, g)$. Similarly, $f \circ h^{-1} = g$ shows that $(C, g) \leq (A, f)$. Thus, the subobjects are isomorphic. □

6.25 COROLLARY

$\approx$ is an equivalence relation on the class of all subobjects of any $\mathscr{C}$-object B. □

6.26 Because of 6.25, we know that the class of all subobjects of an object B is partitioned into equivalence classes of isomorphic subobjects. Thus, via the Axiom of Choice (1.2(4)) for every $\mathscr{C}$-object B there exists a system of representatives for the equivalence relation $\approx$ on the class of all subobjects of B. Such a system of representatives will be called a **representative class of subobjects** of B.

6.27 DEFINITION

A category $\mathscr{C}$ is said to be **well-powered** provided that each $\mathscr{C}$-object has a representative class of subobjects that is a *set*.

DUAL NOTION: **co-(well-powered).** [I.e., every object has a representative class of quotient objects which is a set.]

6.28 EXAMPLES

(1) The categories **Set**, **Grp**, **Top**, $\mathbf{Top_2}$, $\mathbf{BanSp_1}$, and $\mathbf{BanSp_2}$ are well-powered and co-(well-powered).

(2) The partially-ordered class of all ordinal numbers considered as a category (3.5(6)) is well-powered but not co-(well-powered).

Notice that to say that a category is well-powered is equivalent to saying that for each $\mathscr{C}$-object B, there can be only a set $(X_i)_I$ of pairwise non-isomorphic $\mathscr{C}$-objects such that for each i there is some monomorphism $f_i: X_i \to B$. This is so because in any category $\mathscr{C}$, there is only a set of morphisms between any pair of objects.

EXERCISES

6A. Show that in the category of commutative cancellative semigroups, $A \xrightarrow{f} B$ is an epimorphism if and only if for all $b \in B$ there exist $a_1, a_2 \in A$ such that $f(a_1) + b = f(a_2)$. [*Hint:* Embed each commutative cancellative semigroup in an abelian group. There the epimorphisms are surjections.]

6B. For any $\mathscr{C}$-morphism $A \xrightarrow{f} B$, let

$$\bar{f}: \bigcup \{hom(X, A) \mid X \in Ob(\mathscr{C})\} \to \bigcup \{hom(X, B) \mid X \in Ob(\mathscr{C})\}$$

be defined by: $\bar{f}(g) = f \circ g$; and let

$$\underline{f}: \bigcup \{hom(B, X) \mid X \in Ob(\mathscr{C})\} \to \bigcup \{hom(A, X) \mid X \in Ob(\mathscr{C})\}$$

be defined by: $\underline{f}(g) = g \circ f$.

Prove that f is a $\mathscr{C}$-monomorphism if and only if $\bar{f}$ is an injective function and that f is a $\mathscr{C}$-epimorphism if and only if $\underline{f}$ is an injective function.

6C. Let $\mathscr{B}$ be a (full) subcategory of $\mathscr{C}$.
(a) Show that a $\mathscr{B}$-monomorphism (resp. $\mathscr{B}$-epimorphism, $\mathscr{B}$-bimorphism) is not necessarily a $\mathscr{C}$-monomorphism (resp. $\mathscr{C}$-epimorphism, $\mathscr{C}$-bimorphism).
(b) Prove that every $\mathscr{B}$-morphism that is a $\mathscr{C}$-monomorphism (resp. $\mathscr{C}$-epimorphism, $\mathscr{C}$-bimorphism) is necessarily a $\mathscr{B}$-monomorphism (resp. $\mathscr{B}$-epimorphism, $\mathscr{B}$-bimorphism).
(c) Compare these facts with those of Exercise 5F.

6D. Let $\tilde{\mathscr{C}}$ be a quotient category of $\mathscr{C}$.
(a) If f is a $\mathscr{C}$-monomorphism (resp. $\mathscr{C}$-epimorphism, $\mathscr{C}$-bimorphism) then must $\tilde{f}$ be a $\tilde{\mathscr{C}}$-monomorphism (resp. $\tilde{\mathscr{C}}$- epimorphism, $\tilde{\mathscr{C}}$-bimorphism)?
(b) If $\tilde{f}$ is a $\tilde{\mathscr{C}}$-monomorphism (resp. $\tilde{\mathscr{C}}$-epimorphism, $\tilde{\mathscr{C}}$-bimorphism) then must f be a $\mathscr{C}$-monomorphism (resp. $\mathscr{C}$-epimorphism, $\mathscr{C}$-bimorphism)?

6E. Prove that if (f, g) is a monomorphism in the arrow category $\mathscr{C}^2$, then f is a monomorphism in $\mathscr{C}$.

6F. Prove that if f is a $\mathscr{C}$-epimorphism and $g \circ f$ is a $\mathscr{C}$-section, then g is a $\mathscr{C}$-section.

6G. Form the dual of 6F.

6H. *Group Morphisms*
(a) Show that if K is a subgroup of the (finite) group H, then there exists a (finite) group G and group homomorphisms $f_1, f_2: H \to G$ such that

$$K = \{h \in H \mid f_1(h) = f_2(h)\}.$$

[*Hint:* Consider the set X obtained from the set $\{hK \mid h \in H\}$ of all left K-cosets of H, by adjoining a single new element $\hat{K}$. Let G be the permutation group of X, and let $\rho: X \to X$ be the permutation which interchanges the elements eK $(= K)$ and $\hat{K}$, and leaves all other elements of X fixed.

Define $f_1, f_2: H \to G$ by:

$$f_1(h)(S) = \begin{cases} hh'K & \text{if } S = h'K \\ \hat{K} & \text{if } S = \hat{K} \end{cases}$$

$$f_2(h) = \rho \circ f_1(h) \circ \rho^{-1}.]$$

(b) Use part (a) to show that the epimorphisms in **Grp** are precisely the surjective homomorphisms; likewise in the category of finite groups.

6I. *Induction Algebras*
An **induction algebra** is a triple $(A, 0_A, f_A)$ where A is a set, $0_A \in A$ and $f_A: A \to A$ is a function such that the following holds:

If $D \subset A$ and $0_A \in D$ and $f_A[D] \subset D$, then $D = A$.

A **homomorphism** $(A, 0_A, f_A) \xrightarrow{g} (B, 0_B, f_B)$ between induction algebras is a function $g: A \to B$ such that $g(0_A) = 0_B$ and $g(f_A(a)) = f_B(g(a))$ for each $a \in A$.
(a) Show that in the category **IndAlg** of induction algebras and homomorphisms between them, there are monomorphisms that are not injective functions on the underlying sets.
(b) Show that **IndAlg** is a quasi-ordered class (3.5(6)), so that each of its morphisms is a bimorphism.

6J. Show that if the monoid of natural numbers under addition is considered as a category (3.5(7)), then every morphism is a bimorphism, but zero is the only isomorphism.

6K. Prove that if $\mathscr{C}$ is a category such that every $\mathscr{C}$-epimorphism is a $\mathscr{C}$-retraction, then $\mathscr{C}$ is balanced.

6L. Form the dual of Exercise 6K.

6M. Show that in a category $\mathscr{C}$, it is possible for (X, f) and (Y, g) to be non-isomorphic subobjects of an object Z, even though X and Y are $\mathscr{C}$-isomorphic objects.

6N. Form the dual statements of Proposition 6.24 and Corollary 6.25.

·6O. Prove that the category of sets and relations (3.5(2)) is balanced.

§7 INITIAL, TERMINAL, AND ZERO OBJECTS

Initial Objects

7.1 MOTIVATING PROPOSITION

The set $\varnothing$ has the property that for every set B, there exists one and only one function from $\varnothing$ to B.

Proof: It is the empty function from $\varnothing$ to B. □

7.2 DEFINITION

An object X in a category $\mathscr{C}$ is called an **initial object for $\mathscr{C}$** (or a **$\mathscr{C}$-initial object**) provided that for all $\mathscr{C}$-objects B, $hom_{\mathscr{C}}(X, B)$ has exactly one member.

7.3 EXAMPLES

(1) Each of **Set**, **SGrp**, and **Top** has a unique initial object (the empty set, the empty semigroup and the empty space).
(2) **Mon**, **Grp**, **Ab**, and R-**Mod** each have initial objects (the trivial monoids, groups and R-modules).
(3) The ring **Z** of integers is an initial object in **Rng**.

(4) **BooAlg** has initial objects (the two-element boolean algebras).

(5) **Field** has no initial object.

(6) A quasi-ordered class considered as a category has an initial object if and only if it has a smallest member.

7.4 PROPOSITION

Any two $\mathscr{C}$-initial objects, X and Y, are isomorphic.

Proof: Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ be the morphisms guaranteed by the definition. By uniqueness

$$X \xrightarrow{g \circ f} X = X \xrightarrow{1_X} X \qquad \text{and} \qquad Y \xrightarrow{f \circ g} Y = Y \xrightarrow{1_Y} Y.$$

Thus, f and g are sections and retractions, so isomorphisms. □

Terminal Objects

7.5 DEFINITION

An object X in a category $\mathscr{C}$ is called a **terminal object for** $\mathscr{C}$ (or a $\mathscr{C}$-**terminal object**) provided that for all objects B in $\mathscr{C}$, $hom_{\mathscr{C}}(B, X)$ has exactly one member.

7.6 EXAMPLES

(1) Each of **Set**, **SGrp**, **Mon**, **Grp**, **Ab**, R-**Mod**, **Rng**, **Top**, **LinTop**, and **BooAlg**† has terminal objects (the "singletons").

(2) **Field**†† has no terminal objects.

(3) A quasi-ordered class considered as a category has a terminal object if and only if it has a largest member.

7.7 PROPOSITION

Initial object and terminal object are dual concepts. □

7.8 PROPOSITION

Any two $\mathscr{C}$-terminal objects are isomorphic.

Proof: Dualize Proposition 7.4. □

Zero Objects

7.9 DEFINITION

A $\mathscr{C}$-object is called a **zero object for** $\mathscr{C}$ (or a $\mathscr{C}$-**zero object**) provided that it is both a $\mathscr{C}$-initial object and a $\mathscr{C}$-terminal object.

7.10 EXAMPLES

(1) **Grp**, **Mon**, **Ab**, R-**Mod**, **TopGrp**, **LinTop**, $\mathbf{BanSp_1}$, $\mathbf{BanSp_2}$, **pSet**, and **pTop** have zero objects.

(2) **Set**, **Top**, **SGrp**, **Rng**, R-**Alg**, **BooAlg**, **POS**, and **Lat** do not have zero objects.

† For boolean algebras, we do *not* require that $0 \neq 1$.
†† Recall that for each field, $0 \neq 1$.

7.11 PROPOSITION

Any two $\mathscr{C}$-zero objects are isomorphic. □

EXERCISES

7A. Determine the initial, terminal and zero objects (when they exist) of the categories given in Examples 2.2 and 3.5.

7B. Prove that if X is a $\mathscr{C}$-initial (resp. $\mathscr{C}$-terminal, $\mathscr{C}$-zero) object, then

$$hom_{\mathscr{C}}(X, X) = \{1_X\}.$$

7C. Let $X \xrightarrow{f} A$ be a $\mathscr{C}$-morphism.

(a) Prove that if X is a terminal object, then f is a monomorphism.
(b) Prove that if $\mathscr{C}$ is connected and X is an initial object, then f is a monomorphism.
(c) Show that (b) is false if the condition that $\mathscr{C}$ is connected is deleted.

7D. Prove that if X is a $\mathscr{C}$-initial object and Y is a $\mathscr{C}$-terminal object, then the following are equivalent:

(a) $\mathscr{C}$ has a zero object.
(b) X and Y are isomorphic.
(c) $hom_{\mathscr{C}}(Y, X) \neq \emptyset$.
(d) $\mathscr{C}$ is connected.

7E. Let $\mathscr{C}$ be a category with an initial object. Prove that (f, g) is a monomorphism in the arrow category $\mathscr{C}^2$ if and only if both f and g are monomorphisms in $\mathscr{C}$. (Thus, for example, (f, g) is a monomorphism in **TopBun** if and only if both f and g are injective.)

§8 CONSTANT MORPHISMS, ZERO MORPHISMS, AND POINTED CATEGORIES

8.1 MOTIVATING PROPOSITION

If $f: A \to B$ is a function from a non-empty set A to the set B, then the following are equivalent:

(1) f is a constant function, i.e., $f[A]$ is a singleton.
(2) For all sets C and for all functions $r, s: C \to A$, $f \circ r = f \circ s$.
(3) f can be "factored through" a singleton set. □

8.2 DEFINITION

A $\mathscr{C}$-morphism $A \xrightarrow{f} B$ is said to be

(1) A **constant morphism in** $\mathscr{C}$ (or a $\mathscr{C}$**-constant morphism**) provided that for each $C \in Ob(\mathscr{C})$ and for all $r, s \in hom_{\mathscr{C}}(C, A)$, $f \circ r = f \circ s$.

(2) A **coconstant morphism in** $\mathscr{C}$ (or a $\mathscr{C}$**-coconstant morphism**) provided that f is a constant morphism in $\mathscr{C}^{op}$ (i.e., "constant" and "coconstant" are dual notions).

(3) A **zero morphism in** $\mathscr{C}$ (or a $\mathscr{C}$-**zero morphism**) provided that it is both a $\mathscr{C}$-constant morphism and a $\mathscr{C}$-coconstant morphism.

8.3 EXAMPLES

(1) In **Set** or **Top** $A \xrightarrow{f} B$ is a constant morphism if and only if $A = \varnothing$ or $f[A]$ is a singleton. The only coconstants in these categories are functions with empty domain; hence, these are the only zero morphisms.

(2) In **Grp**, R-**Mod**, **Mon**, **LinTop**, $\mathbf{BanSp_1}$, or $\mathbf{BanSp_2}$, $A \xrightarrow{f} B$ is a constant morphism (resp. coconstant morphism, zero morphism) if and only if $f[A]$ is the identity element of B.

(3) Let X and Y be distinct infinite sets, $Ob(\mathscr{C}) = \{X, Y\}$, $hom_{\mathscr{C}}(X, X) = \{1_X\}$, $hom_{\mathscr{C}}(Y, Y) = \{1_Y\}$, $hom_{\mathscr{C}}(Y, X) = \varnothing$, and $hom_{\mathscr{C}}(X, Y) = Y^X$. Then every $\mathscr{C}$-morphism from X to Y is simultaneously a bimorphism and a zero morphism.

8.4 PROPOSITION

If f is a $\mathscr{C}$-constant (resp. $\mathscr{C}$-coconstant, $\mathscr{C}$-zero) morphism, then whenever the composition is defined, $h \circ f \circ g$ is also a $\mathscr{C}$-constant (resp. $\mathscr{C}$-coconstant, $\mathscr{C}$-zero) morphism.

Proof: By duality we need only to give the proof for constants. If r and s are $\mathscr{C}$-morphisms with common domain such that $g \circ r$ and $g \circ s$ are defined, then if f is constant, $f \circ (g \circ r) = f \circ (g \circ s)$. Thus, $(h \circ f \circ g) \circ r = (h \circ f \circ g) \circ s$; so that $h \circ f \circ g$ is a constant. □

8.5 PROPOSITION

Let $A \xrightarrow{f} B$ be a $\mathscr{C}$-morphism, and T be a $\mathscr{C}$-terminal object. Then (1) *implies* (2). *If, furthermore, $hom_{\mathscr{C}}(T, A) \neq \varnothing$, then* (1) *and* (2) *are equivalent.*

(1) *f can be factored through T.*

(2) *f is a constant morphism.*

Proof: Suppose that $A \xrightarrow{f} B = A \xrightarrow{g} T \xrightarrow{h} B$. If $r, s\colon C \to A$, then since there is only one morphism from C to T, $g \circ r = g \circ s$. Hence, $h \circ g \circ r = h \circ g \circ s$, so that $f \circ r = f \circ s$. Thus, f is a constant morphism. Let f be a constant morphism and $g \in hom_{\mathscr{C}}(T, A)$. Since T is a terminal object, there is a morphism $u\colon A \to T$. Because f is a constant, we have

$$f = f \circ 1_A = f \circ (g \circ u) = (f \circ g) \circ u.$$

Thus, f can be factored through T. □

8.6 PROPOSITION

If f is a $\mathscr{C}$-morphism and X is a zero object for $\mathscr{C}$, then the following are equivalent:

(1) *f is a zero morphism.*

(2) *f is a constant morphism.*

(3) *f is a coconstant morphism.*

(4) *f can be factored through X.*

Proof: By definition, (1) is equivalent to ((2) and (3)). Also, since $\mathscr{C}$ has a zero object, $\mathscr{C}$ is connected. Hence, since X is a terminal object, (2) is equivalent to (4) (8.5). Likewise, since X is an initial object, (3) is equivalent to (4) (dual of 8.5). □

8.7 LEMMA

If $f, g: A \to B$, where f is a $\mathscr{C}$-constant morphism, g is a $\mathscr{C}$-coconstant morphism and $hom_{\mathscr{C}}(B, A) \neq \varnothing$, then $f = g$.

Proof: Let $h: B \to A$. Then, by the definitions of constant and coconstant morphisms

$$f = f \circ 1_A = f \circ (h \circ g) = (f \circ h) \circ g = 1_B \circ g = g. \quad \square$$

8.8 THEOREM

In any category, $\mathscr{C}$, the following are equivalent:

(1) *For all $A, B \in Ob(\mathscr{C})$, $hom_{\mathscr{C}}(A, B)$ contains a zero morphism.*

(2) *For all $A, B \in Ob(\mathscr{C})$, $hom_{\mathscr{C}}(A, B)$ contains exactly one zero morphism.*

(3) *For all $A, B \in Ob(\mathscr{C})$, $hom_{\mathscr{C}}(A, B)$ contains exactly one constant morphism.*

(4) *For all $A, B \in Ob(\mathscr{C})$, $hom_{\mathscr{C}}(A, B)$ contains exactly one coconstant morphism.*

(5) *For all $A, B \in Ob(\mathscr{C})$, $hom_{\mathscr{C}}(A, B)$ contains at least one constant morphism and at least one coconstant morphism.*

(6) *There exists a "choice function" selecting exactly one element out of each set $hom_{\mathscr{C}}(A, B)$ such that the composition (from the left or the right) of a selected morphism with any morphism is again a selected morphism (if the composition is defined).*

Proof: We will show (1) ⇒ (2) ⇒ (6) ⇒ (5) ⇒ (3) ⇒ (1).
Since (1) is self-dual and (3) is dual to (4), this will imply that all of the conditions are equivalent.

(1) ⇒ (2). Immediate from the lemma, since by (1), $hom_{\mathscr{C}}(B, A) \neq \varnothing$.

(2) ⇒ (6). Let the "selected" morphism be the unique zero morphism. By Proposition 8.4, the composition of a zero morphism with any morphism is a zero morphism.

(6) ⇒ (5). Let $A \xrightarrow{f} B$ be the "selected" morphism, and let $r, s: C \to A$. Then $f \circ r$ and $f \circ s$ are "selected" morphisms in $hom_{\mathscr{C}}(C, A)$. By the uniqueness of selection, $f \circ r = f \circ s$. Hence, f is a constant morphism. By a dual argument, f is a coconstant morphism.

(5) ⇒ (3). Let $f, g \in hom_{\mathscr{C}}(A, B)$ be constant morphisms. By (5), $\mathscr{C}$ is connected and there is a coconstant morphism $h: A \to B$. Hence, by the lemma, $f = h$ and $g = h$.

(3) ⇒ (1). Let $f: A \to B$ be a constant morphism. If $B \underset{s}{\overset{r}{\rightrightarrows}} C$ is a pair of $\mathscr{C}$-morphisms, then $r \circ f$ and $s \circ f$ are constant morphisms from A to C and consequently are identical. Hence, f is a coconstant and so is a zero morphism. □

8.9 DEFINITION

A category $\mathscr{C}$ is said to be **pointed** provided that it satisfies one of the equivalent conditions of Theorem 8.8.

8.10 PROPOSITION

(1) *Every category which has a zero object is pointed.*

(2) *Every full subcategory of a pointed category is pointed.* □

As it turns out, the pointed categories are "essentially" the full subcategories of categories with a zero object (see Exercise 12F).

8.11 EXAMPLES

(1) **Grp**, R-**Mod**, **Mon**, **LinTop**, **pSet**, **pTop**, and the category of infinite groups are pointed.

(2) **Set**, **Top**, **SGrp**, **POS**, **Lat**, and the category of bipointed sets are not pointed.

EXERCISES

8A. Show that in a concrete category $\mathscr{C}$, every morphism that is a constant function on the underlying sets is a constant morphism in $\mathscr{C}$, but that the converse need not be true.

8B. Prove that $A \xrightarrow{f} B$ is a coconstant morphism in **BooAlg** if and only if $f[A]$ is a boolean algebra with one or two members.

8C. Suppose that $A \xrightarrow{f} B \xrightarrow{g} C$ are $\mathscr{C}$-morphisms.
(a) Show that if g is a monomorphism and $g \circ f$ is a constant morphism, then f is a constant morphism.
(b) Show that if $\mathscr{C}$ is pointed, g is a monomorphism, and $g \circ f$ is a zero morphism, then f is a zero morphism.
(c) Form the duals of (a) and (b).

8D. Prove that in a connected category the following are equivalent:
(a) There exists a constant monomorphism with domain A.
(b) Every morphism with domain A is a constant monomorphism.
(c) A is a terminal object.

8E. Let $\mathscr{C}$ be connected and $Z \xrightarrow{h} X \underset{g}{\overset{f}{\rightrightarrows}} Y$. Prove that if f and g are $\mathscr{C}$-constant morphisms such that $f \neq g$, then $f \circ h \neq g \circ h$.

8F. Show that if $X \xrightarrow{f} Y$ is a constant morphism in a connected category, then there exists a unique constant morphism $Y \xrightarrow{h} Y$ such that $h \circ f = f$.

8G. Establish the fact that if $\mathscr{C}$ is a connected category and $W, X, Y \in Ob(\mathscr{C})$, then there exists a one-to-one correspondence between the collection of $\mathscr{C}$-constant morphisms in $hom_{\mathscr{C}}(W, Y)$ and the collection of $\mathscr{C}$-constant morphisms in $hom_{\mathscr{C}}(X, Y)$.

8H. Form the dual of Proposition 8.5.

8I. Prove that in a pointed category, the "choice function" of 8.8(6) is uniquely determined and that it selects exactly the zero morphisms.

8J. If $\mathscr{C}$ is a pointed category, prove that the following are equivalent:

(a) A is a zero object for $\mathscr{C}$.

(b) $hom_{\mathscr{C}}(A, A) = \{1_A\}$.

8K. Prove that "$\mathscr{C}$ is pointed" is a self-dual statement.