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Convexity Theories II.
The Hahn-Banach Theorem for Real Convexity Theories

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Convexity is the appropriate framework for theorems of Hahn-Banach type. Canonical generalizations of the Theorem of Hahn-Banach can be proved for Γ-convex modules for a positive or a real symmetric convexity theory Γ. For the Hahn-Banach Extension Theorem for Γ-convex modules, for positive Γ, the notion of p-continuity, p a Γ-convex functional, is essential.

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There are many different versions and generalizations of the classical Hahn-Banach Theorem. In 1978, Rodé proved in [7] an, what he called, abstract version of the Theorem of Hahn-Banach, which contained all previously proved results of Hahn-Banach type as special cases. The surprising feature in Rodé's proof is the fact that he proves his theorem by using as a foundation the notion of an abstract, equationally defined, universal algebra, which is commutative, i.e. for which all operations commute pairwise.

It is well known that all versions of the Hahn-Banach Theorem require some sort of local convexity. If a topological vector space is not locally convex, then there may be no non-trivial, continuous, linear functional on it. In [1], Day showed that this is e.g. the case for the p-Banach spaces L^p for $0 < p < 1$. Hence, in view of Rodé's result, it is natural to look for commutative, universal algebras, which are in some sense convex, in order to have an optimal framework for Hahn-Banach type theorems. Universal algebras of this type, *totally convex algebras*, were introduced in [6] independently from Hahn-Banach considerations. In [4] it was shown that Rodé's idea led to a remarkable Hahn-Banach Theorem for these algebras. In some aspects, Rodé's proof had to be modified, but, with respect to convexity, became even more canonical.

In recent years several other types of "*convex*" algebras have been introduced and investigated, e.g. positively convex spaces, strictly positively convex spaces, convex and superconvex spaces (cp. [5],[9],[10],[11],[12]). In [10] it is shown that there are uncountably many (canonical) types of "*convex*" algebras. For all these types a Hahn-Banach Theorem would be very useful. Moreover, an analysis of the proof in [4] showed, that it was actually a proof for positively convex spaces, an observation, which led to the proof of the Hahn-Banach Theorem for positively convex spaces in [12]. In the following we will show that a Hahn-Banach Theorem holds for any *positive convexity theory* Γ in the sense of [9], which constitutes a considerable strengthening of the previous results, and finally we will prove a Hahn-Banach Theorem for a real symmetric convexity theory.

It should be pointed out that the introduction of totally and positively convex spaces ([5], [6]) and, more generally, of Γ- convex modules ([9], [10]), i.e. of those theories, which are the appropriate framework for canonical generalizations of the Hahn-Banach Theorem, is a direct consequence of a

This work is an original contribution and will not appear elsewhere.

famous result in category theory. In 1965 Eilenberg and Moore introduced a construction in [2], which later on led to the notion of so-called *Eilenberg-Moore algebras* (cp. [3]), which, for quite a number of mathematical theories, describe what one could call their *algebraic components*. The category of totally convex spaces is the category of Eilenberg-Moore algebras for the theory of Banach spaces and linear contractions and describes the algebraic contents of the latter. Positively convex spaces play the same role for the theory of regularly ordered Banach spaces and positive linear contractions.

Let Γ be a *convexity theory* (in the sense of [9]) over the normed (resp. Banach) semi-ring R. Then Γ consists of sequences $\alpha_* = (\alpha_i \mid i \in \mathbb{N})$ with $\sum_i \|\alpha_i\| \leq 1$, contains the "unit vectors" $\delta^i_* = (\delta^i_k \mid k \in \mathbb{N})$ and is closed under "substitution", i.e. $\langle \alpha_\square . \beta^\square_* \rangle := (\sum_i \alpha_i \beta^i_k \mid k \in \mathbb{N}) \in \Gamma$ for $\alpha_*, \beta^i_* \in \Gamma, i \in \mathbb{N}$. Γ is called *positive*, if, for all $\alpha_* \in \Gamma$ and all $i \in \mathbb{N}, \alpha_i \in \mathbb{R}$ and $\alpha_i \geq 0$ holds. Since the *semi-ring of definition* of Γ, i.e. the sub semi-ring of R generated by $\{\alpha_i \mid \alpha_* \in \Gamma, i \in \mathbb{N}\}$ (cp. [9]), is contained in $\mathbb{R}^+ := \{x \mid x \in \mathbb{R}, x \geq 0\}$, we may assume $R = \mathbb{R}^+$. Γ is said to be a *convexity theory with zero*, if $0_* := (0 \mid i \in \mathbb{N}) \in \Gamma$ holds.

Following [9] we have the concept of a Γ-*convex module*, which is a (non-empty) Γ-algebra X, i.e. a set, on which the elements of Γ operate, and which fulfills two sets of equations

$$(\Gamma C1) \qquad \sum_{i=1}^{\infty} \delta^k_i x^i = x^k,$$

$$(\Gamma C2) \qquad \sum_{i=1}^{\infty} \alpha_i \Big(\sum_{k=1}^{\infty} \beta^i_k x^k\Big) = \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{\infty} \alpha_i \beta^i_k\Big) x^k,$$

for $\alpha_*, \beta^i_* \in \Gamma, x^* = (x^k \mid k \in \mathbb{N}) \in X^{\mathbb{N}}$. The above sums are a convenient way to denote the action of $\alpha_* \in \Gamma$ on the Γ-algebra X, i.e. they are defined by the operation $\alpha_*(X) : X^{\mathbb{N}} \longrightarrow X$:

$$\sum_{i=1}^{\infty} \alpha_i x^i := \alpha_*(X)(x^*).$$

Sometimes, the shorter notation

$$\langle \alpha_* . x^* \rangle := \sum_{i=1}^{\infty} \alpha_i x^i$$

is used, which stresses the *"bilinearity"* of the mapping $\langle \square, \square \rangle : \Gamma \times X^{\mathbb{N}} \longrightarrow X$. It should be noted that in previous papers on "convex" algebras, e.g. [4], [5], [6], [11], [12], the term Γ–convex *"space"* was used instead of *"module"*.

For a positive convexity theory Γ, $\hat{O}(\mathbb{R}) = \{x \mid x \in \mathbb{R}, |x| \leq 1\}$ is a Γ- convex module with Γ operating canonically on $\hat{O}(\mathbb{R})$, i.e. the formal sum $\sum_i \alpha_i x^i$ has its usual meaning in $\mathbb{R}$. In general, $\hat{O}(\mathbb{R})$ will have numerous Γ- convex submodules. One of them is $S(\Gamma) := \{\rho \mid \rho \in \mathbb{R}$ and $(\rho\delta^1_i \mid i \in \mathbb{N}) \in \Gamma\}$, the *set* of *scalars* of Γ. One verifies easily that, for every Γ- convex submodule H of $\hat{O}(\mathbb{R})$, the closure $\overline{H}$ (in the usual sense) of H in $\mathbb{R}$ is again a Γ- convex submodule of $\hat{O}(\mathbb{R})$.

If Γ is a convexity theory with zero and X is a Γ- convex module, then $0_X := \langle 0_* . x^* \rangle$ is independent of $x^* \in X^{\mathbb{N}}$ and called the *zero* of X (cf. [6], (2.4), [10]). It follows easily from the axioms in [9] that a convexity theory with zero has the following property: If $\alpha_* \in \Gamma$ and $T \subset \mathbb{N}$, then $\alpha^T_i := \alpha_i$ for $i \in T$ and $\alpha^T_i := 0$ else, defines a sequence $\alpha^T_* \in \Gamma$. Analogously, for $x^* \in X^{\mathbb{N}}$, we define x^{*T} by $x^{iT} := x^i$ for $i \in T$ and $x^{iT} := 0_X$ else.

(1) DEFINITION: Let Γ be a positive convexity theory, X a Γ- convex module and H a Γ- convex submodule of $\hat{O}(\mathbb{R})$. Then a mapping $p : X \longrightarrow H$ is called

(i) Γ-*convex*, if $p\left(\sum_i \alpha_i x^i\right) \leq \sum_i \alpha_i p(x^i)$, $\alpha_\bullet \in \Gamma$, $x^\bullet \in X^{\mathbb{N}}$.
(ii) Γ-*concave*, if $p\left(\sum_i \alpha_i x^i\right) \geq \sum_i \alpha_i p(x^i)$, $\alpha_\bullet \in \Gamma$, $x^\bullet \in X^{\mathbb{N}}$.
(iii) Γ-*homogeneous*, if $p(\rho x) = \rho p(x)$, $\rho \in S(\Gamma)$, $x \in X$.
(iv) Γ-*sublinear*, if it is Γ-convex and Γ-homogeneous.

We need the following result, that first appeared in [12], (7.1).

(2) LEMMA: *Let Γ be a convexity theory and let X be a Γ-convex module. Let furthermore be $\alpha_\bullet, \beta^i_\bullet \in \Gamma$, and $z^{ij} \in X$. Suppose that two set mappings $f, g : \mathbb{N} \longrightarrow \mathbb{N}$ induce a bijection $(f,g) : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$. Then $(\alpha_{f(i)}\beta^{f(i)}_{f(i)} \mid i \in \mathbb{N}) \in \Gamma$ and*

$$\sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} \beta^i_j z^{ij} = \sum_{k=1}^{\infty} \alpha_{f(k)} \beta^{f(k)}_{g(k)} z^{f(k)g(k)}.$$

Proof: The simple proof in [12] carries over to the general case (see also [10]).

If X is a Γ- convex module, the set of mappings from X to $\hat{O}(\mathbb{R})$ is ordered by the pointwise order, i.e. for $f, g : X \longrightarrow \hat{O}(\mathbb{R})$ we put $f \leq g$, if $f(x) \leq g(x)$ for all $x \in X$. This is the order used in all subsequent statements.

(3) THEOREM (HAHN-BANACH): *Let Γ be a positive convexity theory, X a Γ- convex module and H a connected, closed, Γ- convex submodule of $\hat{O}(\mathbb{R})$. If $q : X \longrightarrow H$ is Γ- concave, $p : X \longrightarrow H$ is Γ- convex and $q \leq p$ holds, then there exists a morphism $f : X \longrightarrow H$ of Γ- convex modules with $q \leq f \leq p$.*

Proof: The proof proceeds in the same way as the proof of (2.2) in [4], with some modifications. One shows that the set $\{\pi \mid \pi : X \longrightarrow H, \pi\ \Gamma\text{-convex and } q \leq \pi \leq p\}$ is inductively ordered from below. For this, one needs that H is closed and connected. Let $f : X \longrightarrow H$ denote a minimal element of this set. Then f turns out to be Γ-homogeneous. Next, one shows that, for $n \in \mathbb{N}, \alpha_\bullet \in \Gamma, x^\bullet \in X^{\mathbb{N}}$

$$f(\langle \alpha_\bullet, x^\bullet \rangle) \geq \sum_{i=1}^{n} \alpha_i f(x_i) + \sum_{i=n+1}^{\infty} \alpha_i q(x_i)$$

holds. As in [4], (2.2), this is proved by induction. One puts, for $\alpha_{n+1} > 0$,

$\pi(z) := inf\{\ \alpha_{n+1}^{-1}(f(\langle \alpha_\bullet, t^\bullet \rangle) - \sum_{i=1}^{n} \alpha_i f(t_i) - \sum_{i=n+2}^{\infty} \alpha_i q(t_i)) \mid t^\bullet \in X^{\mathbb{N}},\ t_{n+1} = z,\ t_i = \rho_i x_i$ with $\rho_i \in S(\Gamma)$ for $1 \leq i \leq n\ \}$.

In order to show that this defines a mapping $\pi : X \longrightarrow H$ one again needs that H is closed. One also uses the fact that $S(\Gamma)$ is a Γ- convex module (cf. [10], (1.2)).

The next statement is a generalization of (7.2) in [12] and its proof follows the one given there.

(4) THEOREM: *Let Γ be a positive convexity theory with zero. Let furthermore X be a Γ- convex module, X_0 a Γ- convex submodule of X and H a Γ- convex submodule of $\hat{O}(\mathbb{R})$. Suppose that $p : X \longrightarrow H$ is Γ- convex and $r : X_0 \longrightarrow H$ Γ-concave, such that*

(a) *if $\alpha_\bullet \in \Gamma, x^\bullet \in X^{\mathbb{N}}$* and $T_i = \{i \mid i \in \mathbb{N},\ x^i \in X_0\}$, *then*

$$r(\langle \alpha^T_\bullet, x^{\bullet T} \rangle) \leq p(\langle \alpha_\bullet, x^\bullet \rangle).$$

(b) *if* $\langle\alpha_\bullet, x^\bullet\rangle \in X_0$ *for* $\alpha_\bullet \in \Gamma, x^\bullet \in X^{\mathbb{N}}$, *and* $T := \{i \mid i \in \mathbb{N},\ x^i \in X_0\}$, *then*

$$r(\langle\alpha_\bullet^T, x^{\bullet T}\rangle) \le r(\langle\alpha_\bullet, x^\bullet\rangle).$$

Then there exists a mapping $q : X \longrightarrow \overline{H}$ *satisfying*

(i) q *is* Γ - *concave,*
(ii) $q/X_0 = r$,
(iii) $q \le p$.

Proof: Define $q_0 : X \longrightarrow \mathbb{R}$ by $q_0/X_0 := r$ and $q_0(X \setminus X_0) := \{0\}$. Put

$$q(x) := sup\{\ \sum_{i=1}^{\infty} \alpha_i q_0(x^i) \mid \alpha_\bullet \in \Gamma, x^\bullet \in X^{\mathbb{N}}, x = \sum_{i=1}^{\infty} \alpha_i x^i\ \}.$$

Clearly, q maps X to $\overline{H}$.

(i): Let $x = \langle\alpha_\bullet, z^\bullet\rangle$, $\alpha_\bullet \in \Gamma, z^\bullet \in X^{\mathbb{N}}$. Let furthermore be $z^i = \sum_{j=1}^{\infty} \beta_j^i z^{ij}$ with $\beta_\bullet^i \in \Gamma$, $z^{ij} \in X$, $i, j \in \mathbb{N}$, arbitrary.
Then, by (2), we have

$$x = \sum_{i=1}^{\infty} \alpha_i z^i = \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} \beta_j^i z^{ij} = \sum_{k=1}^{\infty} \alpha_{f(k)} \beta_{g(k)}^{f(k)} z^{f(k),g(k)},$$

hence, by (2) again,

$$q(x) \ge \sum_{k=1}^{\infty} \alpha_{f(k)} \beta_{g(k)}^{f(k)} q_0(z^{f(k),g(k)}) = \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} \beta_j^i q_0(z^{ij}).$$

Since Γ is positive, we obtain, by passing to the supremum on the right side,

$$q(x) \ge \sum_{i=1}^{\infty} \alpha_i q(z^i).$$

This shows that q is Γ-concave.

(ii): Clearly $x_0 \in X_0$ implies $q(x_0) \ge r(x_0)$. Now, for $x_0 \in X_0$, take a representation $x_0 = \langle\alpha_\bullet, x^\bullet\rangle$ with $\alpha_\bullet \in \Gamma, x^\bullet \in X^{\mathbb{N}}$. By (b) we have

$$r(x_0) \ge r(\langle\alpha_\bullet^T, x^{\bullet T}\rangle) \ge \langle\alpha_\bullet^T, r^{\mathbb{N}}(x^{\bullet T})\rangle =$$

$$\langle\alpha_\bullet^T, q_0^{\mathbb{N}}(x^{\bullet T})\rangle = \langle\alpha_\bullet, q_0^{\mathbb{N}}(x^\bullet)\rangle = \sum_{i=1}^{\infty} \alpha_i q_0(x^i),$$

and passage to the supremum yields $r(x_0) \ge q(x_0)$. Hence $q/X_0 = r$. Here and in the following, for a set mapping $h : X \longrightarrow Y, h^{\mathbb{N}} : X^{\mathbb{N}} \longrightarrow Y^{\mathbb{N}}$ is the canonically induced mapping.

(iii): For $x \in X$ take a representation $x = \sum_{i=1}^{\infty} \alpha_i x^i = \langle\alpha_\bullet, x^\bullet\rangle$. Due to (a), (i) and (ii) we get

$$p(x) \ge r(\langle\alpha_\bullet^T, x^{\bullet T}\rangle) = q(\langle\alpha_\bullet^T, x^{\bullet T}\rangle) \ge \langle\alpha_\bullet^T, q_0^{\mathbb{N}}(x^{\bullet T})\rangle = \langle\alpha_\bullet, q_0^{\mathbb{N}}(x^\bullet)\rangle.$$

Passage to the supremum on the right side yields $p(x) \ge q(x)$.

(5) **COROLLARY:** *Suppose that* H *in* (4) *is contained in* $\mathbb{R}^+$. *Then* (4), (a) *and* (b), *are necessary for the existence of a mapping* $q : X \longrightarrow \overline{H}$ *satisfying* (4), (i) - (iii).

Proof: Suppose that q satisfies (i) - (iii) in (4). Denote the restriction of q to X_0 by r. Let $x = \langle \alpha_\bullet, x^\bullet \rangle \in X$ and T as in (4), (a). Since $\overline{H} \subset \mathbb{R}^+$, we obtain

$$p(x) \geq q(x) \geq \langle \alpha_\bullet, q^{\mathbb{N}}(x^\bullet)\rangle \geq \langle \alpha_\bullet^T, q^{\mathbb{N}}(x^{\bullet T})\rangle = \langle \alpha_\bullet^T, r^{\mathbb{N}}(x^{\bullet T})\rangle,$$

which is (4), (a). Next, let $x_0 = \langle \alpha_\bullet, x^\bullet \rangle \in X_0$ and T as in (4), (b). Here, we have

$$r(x_0) = q(x_0) \geq \langle \alpha_\bullet, q^{\mathbb{N}}(x^\bullet)\rangle \geq \langle \alpha_\bullet^T, q^{\mathbb{N}}(x^{\bullet T})\rangle = \langle \alpha_\bullet^T, r^{\mathbb{N}}(x^{\bullet T})\rangle,$$

which is (4), (b).

Combining (3) and (4), we get the

(6) PROPOSITION: *Suppose that Γ is a positive convexity theory with zero and that H is a connected, Γ- convex submodule of $\hat{O}(\mathbb{R})$. Let X be a Γ- convex module and X_0 a Γ- convex submodule of X. If $p : X \longrightarrow H$ is Γ- convex, $r : X_0 \longrightarrow H$ is Γ- concave and* (4), (a), (b), *are fulfilled, then there exists a morphism $f : X \longrightarrow \overline{H}$ of Γ- convex modules with $r \leq f/X_0$ and $f \leq p$.*

Next, let Γ be a positive convexity theory with zero, let X be a Γ- convex module, X_0 a Γ- convex submodule of X and H a Γ- convex submodule of $\hat{O}(\mathbb{R})$. Suppose that $p : X \longrightarrow H$ is a mapping. A mapping $f : X_0 \longrightarrow H$ is called *p-continuous*, if $\langle \alpha_\bullet, x^\bullet \rangle = \langle \beta_\bullet, y^\bullet \rangle$, $\alpha_\bullet, \beta_\bullet \in \Gamma$, $x^\bullet, y^\bullet \in X^{\mathbb{N}}$ implies

$$f(\sum_{i\in T} \alpha_i x^i) \leq f(\sum_{i\in U} \beta_i y^i) + p(\sum_{i\in \mathbb{N}\setminus U} \beta_i y^i),$$

where $T := \{i \mid i \in \mathbb{N}, x^i \in X_0\}, U := \{i \mid i \in \mathbb{N}, y^i \in X_0\}$.

Note, if there is a p - continuous mapping, then p is non-negative on the Γ- convex submodule of X generated by $X \setminus X_0$. Moreover, every p - continuous mapping f satisfies

$$-p(\sum_{i\in\mathbb{N}\setminus T} \alpha_i x^i) \leq f(\sum_{i\in T}\alpha_i x^i) - f(\sum_{i\in U}\beta_i y^i) \leq p(\sum_{i\in\mathbb{N}\setminus U}\beta_i y^i).$$

Under the assumptions made for defining the notion of p-continuity, we denote by $d : X_0 \times X \longrightarrow \overline{H}$ the mapping given by

$$d(z_0, z) := inf\{\, p(\sum_{i\in\mathbb{N}\setminus T} \alpha_i x^i) \mid z = \langle \alpha_\bullet, x^\bullet\rangle, z_0 = \langle \alpha_\bullet^T, x^{\bullet T}\rangle, \text{ for some } \alpha_\bullet \in \Gamma,\ x^\bullet \in X^{\mathbb{N}},\ T \subseteq \mathbb{N}\}.$$

(7) LEMMA:

(i) $d(0, z) \leq p(z), z \in X$.

(ii) $d(z_0, z_0) \leq p(0), z_0 \in X_0$.

(iii) *If p is Γ- convex, then, for any $\alpha_\bullet \in \Gamma, z_0^\bullet \in X_0^{\mathbb{N}}, z^\bullet \in X^{\mathbb{N}}$,*

$$d(\langle \alpha_\bullet, z_0^\bullet \rangle, \langle \alpha_\bullet, z^\bullet \rangle) \leq \sum_{i=1}^{\infty} \alpha_i d(z_0^i, z^i)$$

holds.

Proof: The proof of (i) and (ii) is straightforward. To show (iii), let $\varepsilon > 0$. Then, for every $i \in \mathbb{N}$, there are $\beta_\bullet^i \in \Gamma, z^{i\nu} \in X$ and $T_i \subseteq \mathbb{N}$, s.th.

$$z^i = \sum_{\nu=1}^{\infty} \beta_\nu^i z^{i\nu}\ ,\ z_0^i = \sum_{\nu\in T_i} \beta_\nu^i z^{i\nu},$$

and

$$d(z_0^i, z^i) \le p(\sum_{\nu \in \mathbb{N}\setminus T_i} \beta_\nu^i z^{i\nu}) \le d(z_0^i, z^i) + \varepsilon.$$

Denote $T := \bigcup_{i=1}^{\infty} \{i\} \times T_i$, and, with a bijection $(f, g) : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ as in (2), put $U := (f, g)^{-1}(T) \subseteq \mathbb{N}$. Then, in the notation of (2), we have

$$z = \sum_{k=1}^{\infty} \alpha_{f(k)} \beta_{g(k)}^{f(k)} z^{f(k), g(k)},$$

$$z_0 = \sum_{k \in U} \alpha_{f(k)} \beta_{g(k)}^{f(k)} z^{f(k), g(k)},$$

hence

$$d(z_0, z) \le p(\sum_{k \in \mathbb{N}\setminus U} \alpha_{f(k)} \beta_{g(k)}^{f(k)} z^{f(k), g(k)}) =$$

$$= p(\sum_{i=1}^{\infty} \alpha_i \sum_{\nu \in \mathbb{N}\setminus T_i} \beta_\nu^i z^{i\nu}) \le \sum_{i=1}^{\infty} \alpha_i p(\sum_{\nu \in \mathbb{N}\setminus T_i} \beta_\nu^i z^{i\nu})$$

$$\le \sum_{i=1}^{\infty} \alpha_i (d(z_0^i, z^i) + \varepsilon) \le \sum_{i=1}^{\infty} \alpha_i d(z_0^i, z^i) + \varepsilon,$$

which proves (iii).

We are now in the position to prove an

(8) EXTENSION THEOREM OF HAHN-BANACH: *Let Γ be a positive convexity theory with zero, H be a connected, Γ- convex submodule of $\hat{O}(\mathbb{R})$ and $p : X \longrightarrow H$ a Γ- convex mapping from a Γ- convex module X. Then, for any Γ- convex submodule X_0 of X and any p-continuous morphism $f_0 : X_0 \longrightarrow H$ of Γ- convex modules, there is an extension $f : X \longrightarrow \overline{H}$ with $f \le p$.*

Proof: Define

$$q(x) := inf\{d(x_0, x) + f_0(x_0) \mid x_0 \in X_0\}$$

for $x \in X$. (7), (i), yields

$$q(x) \le d(0, x) + f_0(0) = d(0, x) \le p(x).$$

Due to (7), (ii), we have for $x_0 \in X_0$

$$q(x_0) \le d(x_0, x_0) + f_0(x_0) \le p(0) + f_0(x_0) \le f_0(x_0),$$

as p is Γ-convex.

Next, we will show that q is Γ- convex. For this, let $\alpha_* \in \Gamma$, $x^* \in x^{\mathbb{N}}$. For every $\varepsilon > 0$ there are $x_0^i \in X_0$, such that $q(x^i) \le d(x_0^i, x^i) + f_0(x_0^i) < q(x^i) + \varepsilon, i \in \mathbb{N}$. Hence, due to (7), (iii),

$$\begin{aligned} q(\sum_{i=1}^{\infty} \alpha_i x^i) &\le d(\langle \alpha_*, x_0^* \rangle, \langle \alpha_*, x^* \rangle) + f_0(\langle \alpha_*, x_0^* \rangle) \\ &\le \sum_{i=1}^{\infty} \alpha_i d(x_0^i, x^i) + \sum_{i=1}^{\infty} \alpha_i f_0(x_0^i) \\ &\le \sum_{i=1}^{\infty} \alpha_i q(x^i) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, the Γ- convexity of q follows. Finally, put

$$r(x) := sup\{ f_0(\sum_{i \in T} \alpha_i x^i) \mid x = \langle \alpha_*, x^* \rangle, \langle \alpha_*^T, x^{*T} \rangle \in X_0 \},$$

for $x \in X$. We claim that r is Γ- concave. For this, let $\alpha_\bullet \in \Gamma$, $x^\bullet \in X^{\mathbb{N}}$. Given $\varepsilon > 0$, there are $\beta^i_\bullet \in \Gamma, z^{ij} \in X$, $i, j \in \mathbb{N}$, and $T_i \subseteq \mathbb{N}$, s.th.

$$x^i = \sum_{j=i}^{\infty} \beta^i_j z^{ij}, \quad \sum_{j \in T_i} \beta^i_j z^{ij} \in X_0,$$

and

$$r(x^i) - \varepsilon < f_0(\sum_{j \in T_i} \beta^i_j z^{ij}) \leq r(x^i),$$

$i \in \mathbb{N}$. For $x = \sum_{i=1}^{\infty} \alpha_i x^i$, we get from (2)

$$x = \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} \beta^i_j z^{ij} = \sum_{k=1}^{\infty} \alpha_{f(k)} \beta^{f(k)}_{g(k)} z^{f(k),g(k)}.$$

If we define, for $i, j \in \mathbb{N}$,

$$u^{ij} := \begin{cases} z^{ij} & \text{for } \ j \in T_i, \\ 0, & \text{else,} \end{cases}$$

and $T := \{k \mid k \in \mathbb{N} \text{ and } g(k) \in T_{f(k)}\}$, (2) yields

$$\sum_{k \in T} \alpha_{f(k)} \beta^{f(k)}_{g(k)} z^{f(k),g(k)} = \sum_{k=1}^{\infty} \alpha_{f(k)} \beta^{f(k)}_{g(k)} u^{f(k),g(k)} = \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} \beta^i_j u^{ij} = \sum_{i=1}^{\infty} \alpha_i \sum_{j \in T_i} \beta^i_j z^{ij} \in X_0.$$

Hence

$$r(\sum_{i=1}^{\infty} \alpha_i x^i) \geq f_0(\sum_{i=1}^{\infty} \alpha_i \sum_{j \in T_i} \beta^i_j z^{ij}) = \sum_{i=1}^{\infty} \alpha_{i_0} f_0(\sum_{j \in T_i} \beta^i_j z^{ij}) \geq \sum_{i=1}^{\infty} \alpha_i r(x^i) - \varepsilon,$$

for any $\varepsilon > 0$, i.e. r is Γ-concave. Moreover, the p-continuity of f_0 implies, that, for $x = \langle \alpha_\bullet, x^\bullet \rangle = \langle \beta_\bullet, y^\bullet \rangle$, $T := \{i \mid i \in \mathbb{N}, x^i \in X_0\}, U := \{i \mid i \in \mathbb{N}, y^i \in X_0\}$,

$$f_0(\sum_{i \in T} \alpha_i x^i) \leq f_0(\sum_{i \in U} \beta_i y^i) + p(\sum_{i \in \mathbb{N} \setminus U} \beta_i y^i)$$

holds. If we only regard $\beta_\bullet, y^\bullet$, s.th. $x_0 = \sum_{i \in U} \beta_i y^i$ we get

$$f_0(\sum_{i \in T} \alpha_i x^i) \leq f_0(x_0) + d(x_0, x),$$

resp.

$$r(x) \leq q(x).$$

Now, $r(x) \geq f_0(0) = 0$ for all $x \in X$, and we get

$$0 \leq r(x) \leq q(x) \leq p(x), \ x \in X.$$

As $0 \in H$ and $p(x) \in H$, we get $q(x) \in H$, because H is connected and thus also $r(x) \in H$. Hence (3) implies the existence of a morphism $f : X \longrightarrow \overline{H}$ with $r \leq f \leq q \leq p$. For $x_0 \in X_0$ this implies

$$f_0(x_0) \leq r(x_0) \leq f(x_0) \leq q(x_0) \leq f_0(x_0)$$

i.e. $f(x_0) = f_0(x_0)$; f is an extension of f_0.

One should note that the last section of this proof shows that actually $f(x) \in \overline{H^+}$ for $x \in X$, where $H^+ := H \cap \mathbb{R}^+$, the non-negative part of H is a Γ- convex submodule of H. Hence, if we discuss the possibility of extending $f_0 : X_0 \longrightarrow H$ by a morphism $f : X \longrightarrow \overline{H}$, s.th. $f(x) \geq 0$ for all $x \in X$ and

$f \leq p$, the condition of p-continuity is also necessary. Because for $\langle \alpha_., x^. \rangle = \langle \beta_., y^. \rangle$ with $T, U \subseteq \mathbb{N}$ as in the definition of p-continuity, we get

$$\begin{aligned} f_0\left(\sum_{i\in T}\alpha_i x^i\right) &\leq \sum_{i\in T}\alpha_i f(x^i) + \sum_{i\in \mathbb{N}\setminus T}\alpha_i f(x^i) \\ &= \sum_{i\in U}\beta_i f(y^i) + \sum_{i\in \mathbb{N}\setminus U}\beta_i f(y^i) \\ &\leq f_0\left(\sum_{i\in U}\beta_i y^i\right) + p\left(\sum_{i\in \mathbb{N}\setminus U}\beta_i f(y^i)\right). \end{aligned}$$

In order to get the analogue of the classical extension theorem we need the concept of real symmetric convexity theories. We call a real convexity theory Γ *symmetric*, if the following conditions are fulfilled:

(i) $0_. \in \Gamma$,
(ii) $\alpha_. \in \Gamma$ implies $-\alpha_. = (-\alpha_i \mid i \in \mathbb{N}) \in \Gamma$.
(iii) there is an $\alpha_. \in \Gamma$ with at least two non-zero components .

Let Γ by symmetric and let $\alpha_. \in \Gamma$ have at least two non-zero entries. We may assume $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. It follows easily that $(\alpha_i, \sum_{i=2}^{\infty}\alpha_i, 0, 0 \ldots)$ is in Γ. If $\sum_{i=2}^{\infty}\alpha_i = 0$, replace $\alpha_.$ by $(\alpha_1, -\alpha_2, \alpha_3, \ldots)$, which is again in Γ, and form $(\alpha_1, -\alpha_2 + \sum_{i=3}^{\infty}\alpha_i, 0, 0, \ldots) \in \Gamma$. Then the second component of the latter is different from 0. In any case there is a $(\alpha_1, \alpha_2, 0, 0, \ldots) \in \Gamma$ with $\alpha_1\alpha_2 \neq 0$. Hence, the element

$$\alpha_1(\alpha_2, 0, 0, \ldots) + \alpha_2(0, \pm\alpha_1, 0, \ldots) = (\alpha_1\alpha_2, \pm\alpha_1\alpha_2, 0, 0, \ldots)$$

is also in Γ. In other words, Γ contains an element of the form $(\alpha, \pm\alpha, 0, 0, \ldots)$ with $\alpha \neq 0$.

If Γ is real symmetric and X is a Γ- convex module, one verifies easily that $\sigma : X \longrightarrow X$, defined by $\sigma(x) := -x := \sum_{i=1}^{\infty}(-\delta_i^1)x$ is an automorphism of Γ- convex modules. Finally we put $\Gamma_+ := \{\alpha_. \mid \alpha_. \in \Gamma \text{ and } \alpha_i \geq 0, i \in \mathbb{N}\}$.

(9) **PROPOSITION:** *Let Γ be a real symmetric convexity theory. H a connected Γ- convex submodule of $\hat{O}(\mathbb{R})$ and X a Γ- convex module. Suppose that $p : X \longrightarrow H$ is a Γ_+- convex mapping with $p(0) = 0$. Then there exists a morphism $f : X \longrightarrow \overline{H}$ of Γ- convex modules, such that $-p(-x) \leq f(x) \leq p(x)$ holds for $x \in X$.*

Proof: Put $q(x) := -p(-x)$, then $q : X \longrightarrow H$. Let $(\alpha, -\alpha, 0, 0, \ldots) \in \Gamma$ with $\alpha > 0$. then $\langle \alpha_., x^. \rangle = 0$ for $x^i := x, i \in \mathbb{N}$. In abbreviated form the last equation is mostly written as $\alpha x - \alpha x = \alpha x + \alpha(-x) = 0$. We get

$$0 = p(0) \leq \alpha p(x) + \alpha p(-x)$$

and thus $q(x) = -p(-x) \leq p(x)$. As q is obviously Γ_+- concave, (3) yields the existence of a morphism $f : X \longrightarrow \overline{H}$ of Γ_+- convex modules with $-p(-x) \leq f(x) \leq p(x), x \in X$. As σ is a morphism of Γ- convex modules and commutes with f, we get $f(-x) = -f(x)$ for $x \in X$. For $\alpha_. \in \Gamma$ we denote by $|\alpha|_.$ the sequence defined by $|\alpha|_i := |\alpha_i|$, $i \in \mathbb{N}$. It is easy to check that $|\alpha|_. \in \Gamma_+$. Hence we have

$$\begin{aligned} f\left(\sum_{i=1}^{\infty}\alpha_i x^i\right) &= f\left(\sum_{i=1}^{\infty}|\alpha_i|(sgn(\alpha_i)x^i)\right) \\ &= \sum_{i=1}^{\infty}|\alpha_i| f(sgn(\alpha_i)x^i) = \sum_{i=1}^{\infty}\alpha_i f(x^i), \end{aligned}$$

showing that f is a morphism of Γ- convex modules.

(10) EXTENSION THEOREM OF HAHN-BANACH (Symmetric Version): *Let* Γ *be a real symmetric convexity theory and* H *a connected* Γ- *convex submodule of* $\hat{O}$ (IR). *Suppose that* X *is a* Γ- *convex module and* $p : X \longrightarrow H$ *is* Γ*-sublinear. Then, for every* Γ- *convex submodule* X_0 *of* X, *any morphism* $f_0 : X_0 \longrightarrow H$ *of* Γ- *convex modules with* $f_0 \leq p/X_0$ *can be extended to a morphism* $f : X \longrightarrow \overline{H}$ *of* Γ- *convex modules, such that* $-p(-x) \leq f(x) \leq p(x)$, *for all* $x \in X$.

Proof: There is an $\alpha > 0$, such that $(\alpha, \pm\alpha, 0, 0, \ldots) \in \Gamma$. With this α one defines, as in the proof of (3.2) in [4], for $x \in X$,

$$q(x) := \alpha^{-1} inf\{p(\alpha x - \alpha x_0) + \alpha f_0(x_0) \mid x_0 \in X_0\}$$

and proves in exactly the same way that q is Γ_+- convex and fulfills $-p(-x) \leq q(x) \leq p(x)$. Now the existence of f with $-q(-x) \leq f(x) \leq q(x)$ follows directly from (9). For $x_0 \in X_0$, we have $q(x_0) \leq f_0(x_0)$, which implies $-q(-x_0) \geq f_0(x_0)$. Hence, for $x_0 \in X_0$, $f(x_0) = f_0(x_0)$ holds, i.e. f is an extension of f_0.

It should be noted, that, in the symmetric case, no p-continuity is needed, an an assumption, which is necessary in the positive case.

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