Category Theory at Work
H. Herrlich. H.-E. Porst (eds.)

377-386

Sophus Lie's Fundamental Theorem – Categorical Aspects

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Abstract. The Fundamental Theorems of Sophus Lie assert that the construction of the Lie algebra $\mathrm{L}(G)$ for a given finite-dimensional real Lie group G induces a functor $\mathrm{L}: \mathrm{LGrp} \to \mathrm{LAlg}$, which has a right inverse. Moreover, the functor L admits a factorisation $\mathrm{L}' \circ \mathrm{Res}$, where Res denotes restriction to germs of local Lie groups and L' is an isomorphism of categories. In the semigroup case the assignment $(S, B) \mapsto \mathrm{L}(S, B)$, where (S, B) is a local Lie semigroup and $\mathrm{L}(S, B)$ its tangent wedge, gives rise to a functor $\mathrm{L}: \mathrm{LSgrp} \to \mathrm{LWed}$ from the category of Lie semigroups into the category of Lie wedges. This functor also admits a factorisation $\mathrm{L}' \circ \mathrm{Res}$ through the category of germs of local semigroups and L' has a right inverse. This nice correspondence between the local and the infinitesimal aspects does not extend to the global situation. However, it is possible to establish a functor from the category of pointed cones in Lie algebras into the category of topological semigroups, which is almost a right inverse of $\mathrm{L}: \mathrm{LSgrp} \to \mathrm{LWed}$. These functorial properties depend on the involved construction of the (local) semigroups by means of conal curves.

1 From the global to the infinitesimal

With respect to the Lie theory of groups and semigroups the monographs [1], [2], [7], [8] and [22] are considered as standard references. The question whether the Lie theory of semigroups already occurs (at least implicitly) in the original work of Sophus Lie himself has been discussed in [9]. It is also possible to establish Lie's Fundamental Theorems for local analytic loops [12], but we do not include this aspect.

The categories of finite-dimensional real Lie groups and Lie algebras are denoted by LGrp and LAlg, respectively. For every Lie group G let $\mathrm{L}(G) \cong \mathrm{T}_1 G$ be the Lie algebra of G and $\exp_G: \mathrm{L}(G) \to G$ the exponential function. Lie's First Fundamental Theorem may be stated as follows:

Theorem 1.1. [8, VII.1.3, 3.2-3] *The assignment $G \mapsto \mathrm{L}(G)$ induces a functor $\mathrm{L}: \mathrm{LGrp} \to \mathrm{LAlg}$ and for each morphism $f: G_1 \to G_2$ of Lie groups the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{L}(G_1) & \xrightarrow{\mathrm{L}(f) = df(0)} & \mathrm{L}(G_2) \\ {\scriptstyle \exp_{G_1}} \downarrow & & \downarrow {\scriptstyle \exp_{G_2}} \\ G_1 & \xrightarrow[f]{} & G_2 \end{array}$$

This work is an original contribution and will not appear elsewhere.

Moreover, the exponential function induces a diffeomorphism from a 0-neighborhood in the Lie algebra onto a 1-neighborhood in the corresponding group.

Note that the last result induces the structure of a local group on some 0-neighborhood B in $L(G)$. The Baker-Campbell-Hausdorff-(BCH-)mechanism allows an internal characterization of the local multiplication $*$ on B by means of the Lie algebra structure alone and we call $(B, *)$ a BCH-neighborhood.

The category LSgrp of *Lie semigroups* is defined as follows: The objects are pairs (S, G), which consist of a subsemigroup S of a Lie group G such that the identity element 1 of G belongs to S. The morphisms $f: (S_1, G_1) \to (S_2, G_2)$ are morphisms of Lie groups $f: G_1 \to G_2$ satisfying $f(S_1) \subseteq S_2$.

A *local semigroup* (S, B) in a Lie algebra L consists of a BCH-neighborhood $(B, *)$ of L and a subset S of B with

$$0 \in S \quad \text{and} \quad S * S \cap B \subseteq S.$$

A *morphism of local semigroups* $f: (S_1, B_1) \to (S_2, B_2)$ is an analytic map $f: B_1 \to B_2$ satisfying $f(S_1) \subseteq S_2$ which is also a local homomorphism, i.e. $f(0) = 0$ and $x, y, xy \in B_1$ imply $f(xy) = f(x)f(y) \in B_2$.

In order to develop a convenient description of functorial properties of local semigroups we introduce the following notation: Let (S_1, B_1) and (S_2, B_2) be local semigroups within a Lie algebra L. Define

$$(S_1, B_1) \leq (S_2, B_2) \quad \overset{\text{def}}{\Longleftrightarrow} \quad S_1 \subseteq S_2 \quad \text{and} \quad B_1 \subseteq B_2.$$

An *inverse system of local semigroups* within some fixed Lie algebra $L(\mathcal{S})$ is a collection $\mathcal{S}$ which is downward directed with respect to this relation. The inverse system $\mathcal{S}'$ *refines* $\mathcal{S}$ if and only if for each $(S, B) \in \mathcal{S}$ there exist $(S', B') \in \mathcal{S}'$ with $(S', B') \leq (S, B)$. The inverse systems $\mathcal{S}$ and $\mathcal{S}'$ are *equivalent* if they refine each other. The equivalence class of $\mathcal{S}$ is denoted by $[\mathcal{S}]$. Each inverse system may be interpreted as a diagram over some small category induced by a downward directed index set. Suppose $\mathcal{S}_k$ and $\mathcal{T}_k$ are inverse systems over the same index set J_k $(k = 1, 2)$. Two natural transformations $\eta_k: \mathcal{S}_k \to \mathcal{T}_k$ $(k = 1, 2)$ are *equivalent* if $[\mathcal{S}_1] = [\mathcal{S}_2]$ and $[\mathcal{T}_1] = [\mathcal{T}_2]$ and for all (S_k, B_k) in $\mathcal{S}_k$ and (T_k, C_k) in $\mathcal{T}_k$ $(k = 1, 2)$ with $(S_2, B_2) \hookrightarrow (S_1, B_1)$ and $(T_2, C_2) \hookrightarrow (T_1, C_1)$ the following diagram is always commutative:

$$\begin{array}{ccc} (S_1, B_1) & \xrightarrow{\eta_1} & (T_1, C_1) \\ \uparrow & & \uparrow \\ (S_2, B_2) & \xrightarrow[\eta_2]{} & (T_2, C_2) \end{array}$$

The symbol $[\eta]$ will be used for the corresponding classes.

The category GLocSgrp of *germs of local semigroups* is given by the following data: The objects are equivalence classes $\Gamma = [\mathcal{S}]$ of inverse systems of local semigroups such that for all (S_0, B_0), (S_1, B_1) in $\mathcal{S}$ there exists a 0-neighborhood $B \subseteq L(\mathcal{S})$ satisfying $S_0 \cap B = S_1 \cap B$. The morphisms $[\eta]: [\mathcal{S}] \to [\mathcal{T}]$ are equivalence classes of natural transformations. The category

GLocGrp of *germs of local groups* is defined analogously and the projections $(S, B) \mapsto B$ induce a forgetful functor GLocSgrp $\to$ GLocGrp. The *restriction functor* Res: LSgrp $\to$ GLocSgrp maps each Lie semigroup (S, G) onto the germ

$$\mathrm{Res}(S, G) = [(S' \cap B, B)_{B \in \mathcal{U}(0)}],$$

where $S' = \exp_G^{-1}(S)$ and $\mathcal{U}(0)$ is the 0-neighborhood filter of $\mathrm{L}(G)$. Each morphism $g: (S, G) \to (T, H)$ of Lie semigroups induces a morphism $[\eta]$ of germs by means of 0-neighborhood-bases $\mathcal{V}$ and $\mathcal{W}$ in $\mathrm{L}(G)$ and $\mathrm{L}(H)$, respectively, such that both can be considered as diagrams over the same index set J and for all $j \in J$ the inclusion $g(V_j) \subseteq W_j$ is valid. The natural transformation η is determined by the respective (co-)restrictions of $g: G \to H$.

If A is a subset of some Lie algebra L with $0 \in A$, then $X \in L$ is called *subtangent vector of A (at 0)* provided there exist sequences (r_n) in $\mathbb{R}^+$ and (a_n) in A satisfying

$$0 = \lim_{n\to\infty} a_n \quad \text{and} \quad X = \lim_{n\to\infty} r_n a_n.$$

The set of all subtangent vectors of A is denoted by $\mathrm{L}(A)$. If $1 \in A \subseteq G$ for some Lie group G, the subtangent vectors of A are determined by means of the restriction of the exponential function to some 0-neighborhood $B \subseteq \mathrm{L}(G)$ such that $\exp_G$ induces a diffeomorphism onto the 1-neighborhood $\exp_G(B)$ in G.

Recall that a subset W of a finite-dimensional real Lie algebra L is a *wedge* if the following conditions are satisfied:

$$W + W = W, \quad \mathbb{R}^+ W = W \quad \text{and} \quad \mathrm{cl}_L W = W$$

The subset $\mathrm{H}(W) = W \cap (-W)$ is the *edge* of the wedge and a *(pointed) cone* is a wedge C in L with $\mathrm{H}(C) = \{0\}$. If a wedge W in L satisfies the relations

$$e^{\mathrm{ad}\, x} W = W \quad \text{for all} \quad x \in \mathrm{H}(W),$$

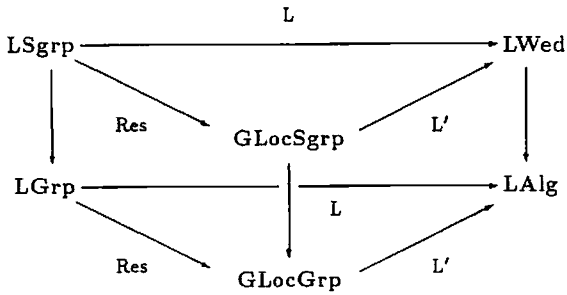
it is called a *Lie wedge*. Obviously every pointed cone is a Lie wedge and in [7, Chapter II–III] various classes of Lie wedges are investigated. The pairs (W, L) consisting of a Lie wedge W in a Lie algebra L are the objects of a category LWed whose morphisms are given by Lie algebra homomorphism $f: L_1 \to L_2$ satisfying $f(W_1) \subseteq W_2$. LCone designates the full subcategory of pointed cones. The following is the semigroup version of Lie's First Fundamental Theorem:

Theorem 1.2. [7, IV.1.27]

(a) *For each Lie semigroup (S, B) the pair $\mathrm{L}(S, B) = (\mathrm{L}(S), \mathrm{L}(G))$ is a Lie wedge and each morphism f of Lie semigroups induces a morphism $\mathrm{L}(f) = df(0)$ of Lie wedges.*

(b) *The functor* L: LSgrp $\to$ LWed *admits a factorisation* $\mathrm{L} = \mathrm{L}' \circ \mathrm{Res}$ *such that for each germ* $\Gamma = [\mathcal{S}]$ *the tangent wedge is determined by* $\mathrm{L}'([\mathcal{S}]) = \mathrm{L}(S, B)$ *for some* (S, B) *in* $\mathcal{S}$.

The relationship between the global, local and infinitesimal properties may be encoded in the following diagram, where vertical arrows always denote forgetful functors induced by the projections $(S, B) \mapsto B$ and $(W, L) \mapsto L$, respectively:



2 From the infinitesimal to the local

According to Lie's Third Fundamental Theorem, for each Lie algebra L there exists a local group $(B, *)$ with $L = \mathrm{L}(B)$. This local group may be realised by some suitably chosen BCH-neighborhood of L (cf. [1, II–III], [8, X] and [22, III.3]). If $\Gamma(L) = [B]$ denotes the germ of local groups induced by B (i. e. $\mathcal{B} = \{B \cap U \mid U \in \mathcal{U}(0)\}$), then the restriction of a morphism $f: L \to L'$ to suitable 0-neighborhoods $B \cap U$ obviously defines a morphism $\Gamma(f): \Gamma(L) \to \Gamma(L')$.

Theorem 2.1. *The functors* L': GLocGrp $\to$ LAlg *and* Γ: LAlg $\to$ GLocGrp *are inverse to each other.*

In the semigroup case the Fundamental Theorem of Lie wedges claims that each Lie wedge (W, L) arises as the tangent wedge of some germ $\Gamma(W, L)$ of local semigroups (cf. [7, IV.8.7], [13, 8.4]). The elements (S, B) of this germ are given by upper sets $S = \uparrow_B 1$ with respect to the W-conal quasi-order on B. Recall that an absolutely continuous curve $\alpha: [0, T_\alpha] \to L$ is a *W-conal curve*, if

$$\alpha'(t) \in d\lambda_{\alpha(t)}(0)(W) \quad \text{for} \quad t \in [0, T_\alpha) \quad \text{a.e.},$$

where $\lambda_x: B \to L$ denotes the left translation by $x \in B$ with respect to the BCH-multiplication. We will say that $x = \alpha(0)$ and $y = \mathrm{end}(\alpha) = \alpha(T_\alpha)$ are *joined by the conal curve* α and express this fact symbolically by $x \prec_V y$ whenever V is an open subset of L with $\mathrm{im}(\alpha) \subseteq V$. We refer to the relation $\prec_V$ as the *(W-)conal quasi-order on* V. Obviously the concept of conal curves can also be transported into the group setting and we will freely use both sides. Since every morphism of Lie wedges preserves conal curves, the assignment $(W, L) \mapsto \Gamma(W, L)$ is functorial:

Theorem 2.2. [23, 2.4.13] *The functor* Γ: LWed $\to$ GLocSgrp *satisfies* $\mathrm{L}' \circ \Gamma = \mathrm{id}_{\mathbf{LWed}}$.

However, Γ is not an inverse of L, since there exist non-equivalent germs of local semigroups with common tangent wedge: Let $L = \mathbb{R}^2$, $S_1 = \mathbb{R}^+ \times \{0\}$ and $S_2 = \{(x, y) \mid x \geq 0, \quad x^2 \geq |y|\}$. Then the tangent wedges of S_1 and S_2 coincide with $W = \mathbb{R}^+ \times \{0\}$, but S_1 and S_2 induce different germs of local semigroups. Examples of this kind motivate the investigation of restricted classes of Lie semigroups, e.g. (strictly) infinitesimally generated semigroups or ray semigroups [7, V].

3 Globality

Every germ $[\mathcal{B}]$ of local Lie groups occurs as the germ of 1-neighborhoods of some global Lie group $\mathrm{E}([\mathcal{B}]) = \tilde{G}$, which may be chosen to be simply connected [8, XII]. The group $\tilde{G}$ can be represented as a subset of the inverse limit of groups G_B indexed by $\mathcal{B}$ [2, II.VII], [22, II.4] and we briefly sketch this construction, since it serves as a prototype of a similar object in the semigroup case. Let $(B, *)$ be a local Lie group (considered as a BCH-neighborhood within some Lie algebra) and denote by G_B the free abstract group over the alphabet B modulo the smallest congruence relation which identifies the word xy with the letter $x * y$ for all letters $x, y \in B$ with $x * y \in B$. The set $\{A \subseteq G_B \mid (\forall a \in A) \quad Aa^{-1} \cap B \in \mathcal{U}(1)\}$ is a basis for a group-topology on G_B such that the canonical maps $\pi_U^V : G_V \to G_U$ for $V \subseteq U$ are covering maps. Let

$$(G, \tau) = \varprojlim\{G_B \mid B \in \mathcal{B}\}$$

be the inverse limit and choose the set of all components of open subsets of G as the basis of a new topology τ' on G. Then the component $\tilde{G}$ of the identity in (G, τ') is simply connected and B can be openly embedded into $\tilde{G}$. By means of standard arguments concerning the extension of local homomorphisms to simply connected groups [2, II.VII], [8, IV.1.7] - or, equivalently, by taking the limits - one obtains the following result:

Theorem 3.1. *The restriction functor* Res: LGrp $\to$ GLocGrp *has a right inverse.*

This convenient situation is not transportable to the case of infinite-dimensional local groups. Counterexamples are discussed e. g. in [1, III.6.Ex.8] and [4, 8]. P. A. Smith [19], W. T. van Est [4], S. Swierczkowski [20] and others have developped cohomological criteria for enlargeability of local groups. Small examples of a non-enlargible abstract local (semi-)groups are given in [21, 6] and [15, 3.6], respectively.

In the semigroup setting the result of (3.1) fails to be true even in the finite-dimensional case. Let us call a Lie wedge (W, L) *analytically global*, if there exists a Lie semigroup (S, G) with $(W, L) = \mathrm{L}(S, G)$. Many necessary and/or sufficient criteria for analytical globality have been investigated (see e.g. [7, VI], [13, VI], [16] and [17]).

Example 3.2. [6],[7, V.4.4ff.] Let H be the 3-dimensional Heisenberg group of real upper triangular (3×3)-matrices with diagonal entries equal to 1. Its Lie algebra $\mathrm{L}(H)$ consists of all (3×3)-matrices of the form

$$[x, y, z] = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The BCH-Multiplikation on $\mathrm{L}(H)$ is given by the equation

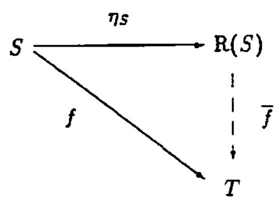
$$[x_1, y_1, z_1] * [x_2, y_2, z_2] = [x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)].$$

Let $a = [1, 0, 0]$, $b = [0, 1, 0]$ and $c = [0, 0, 1]$. The center of $\mathrm{L}(H)$ coincides with $\mathbb{R}c$ and every pointed cone $C \subseteq \mathrm{L}(H)$ which has central points in its interior fails to be analytically global [6]. But there always exists a local semigroup (S, B) in H with $C = \mathrm{L}(S)$, and the semigroup $S' = \langle S \rangle$ generated by

S in the group $(\mathrm{L}(H), *)$ satisfies $S' \cap B \neq S$ for all 0-neighborhoods B. It turns out that $S' = \mathrm{L}(H)$, that is, the local semigroup S badly fails to have a global Lie semigroup extension with the same tangent cone. ∎

By view of the preceding example it is desirable to obtain at least some kind of topological extension of local semigroups. A Lie wedge (W, L) is *topologically global* if there exist a local semigroup (S, B) with $(W, L) = \mathrm{L}(S, B)$ and an open embedding $e: S \to T$ into some topological semigroup preserving all products already defined in S. It has been shown that each pointed cone is topologically global [24, 2.4]. In the subsequent paragraphs we demonstrate some of the functorial properties of this topological approach.

The most general solution of the above mentioned topological embedding problem is given by the relatively free topological semigroup $\mathrm{R}(S)$ over S, which is characterized by the property that for each continuous partial homomorphism $f: S \to T$ into some global topological semigroup T there exists a uniquely determined continuous homomorphism $\overline{f}: \mathrm{R}(S) \to T$ such that the following diagram commutes:



In other words R is left adjoint to the embedding of the category of topological semigroups into the category of topological partial semigroups (cf. [3, Chapter6] or [24, Section 2] for the relevant definitions). The existence of the left adjoint R is a consequence of Freyd's Adjoint Functor Theorem. The fact that there exist local semigroups S with prescribed tangent cone C such that the unit η_S is an open embedding follows from the result that suitable local semigroups can be openly embedded into their one-point-extension S^∞ and the extended multiplication is also associative. This in turn rests on a careful analysis of reachable sets [7, IV], [24, 2.3].

Another approach rests on the concept of conal curves, which has already been very useful for the construction of local semigroups and in the context of analytical globality. Let G be a finite-dimensional Lie group and $C \subset \mathrm{L}(G)$ a pointed cone. It is possible to determine a set $\mathcal{N}(C, G)$ of C-conal curves $\alpha: [0, T_\alpha] \to G$ with $\alpha(0) = 1$ such that $(\mathcal{N}(C, G), \flat)$ is a topological semigroup on a locally compact σ-compact space with respect to concatenation of curves. The main idea is to consider only those conal curves, which are parametrized by a suitable arc-length, and to apply the Ascoli-Theorem (see [24, 3.5] and cf. [18] for a similar approach). If $U \subseteq G$ is an open 1-neighborhood, we will write $\mathcal{N}(C, U) = \{\alpha \in \mathcal{N}(C, G) \mid \mathrm{im}(\alpha) \subseteq U\}$. If k_U denotes the kernel of the restriction of the map $\mathrm{end}: \mathcal{N}(C, G) \to G$ to $\mathcal{N}(C, U)$, i.e.

$$k_U = \{(\alpha, \beta) \in \mathcal{N}(C, U) \mid \mathrm{end}(\alpha) = \mathrm{end}(\beta)\},$$

then the partial semigroups

$$\uparrow_U^{\mathcal{N}} 1 = \{x \in U \mid \exists \alpha \in \mathcal{N}(C, U) \quad \text{with} \quad x = \mathrm{end}(\alpha)\}$$

and $\mathcal{N}(C,U)/k_U$ are isomorphic, where the product of $\alpha, \beta \in \mathcal{N}(C,U)$ is defined and equal to the concatenation $\alpha\flat\beta$ if and only if $\mathrm{im}(\alpha\flat\beta) \subseteq U$. The relation k_U may also be considered as a binary relation on the topological semigroup $\mathcal{N}(C,G)$. Let κ_U be the closed congruence generated by k_U. The quotient semigroup

$$(\mathrm{P}_U(C,G), \natural) = \frac{(\mathcal{N}(C,G), \flat)}{\kappa_U}$$

is called the *conal path semigroup with respect to C and U*. Since $\mathcal{N}(C,G)$ is locally compact and σ-compact, the conal path semigroup is a topological semigroup on a k_ω-space [14, 2.4].

Proposition 3.3. [24, 4.3] *Let G be a Lie group and $C \subset \mathrm{L}(G)$ a pointed cone.*

(a) *There exists a germ $\Gamma = [S]$ of local Lie semigroups in G with $C = \mathrm{L}'(\Gamma)$ and each $(S,B) \in \mathcal{S}$ admits an open embedding $\iota_B: S \to \mathrm{P}_B(C,G)$ into the conal path semigroup $\mathrm{P}_B(C,G)$.*

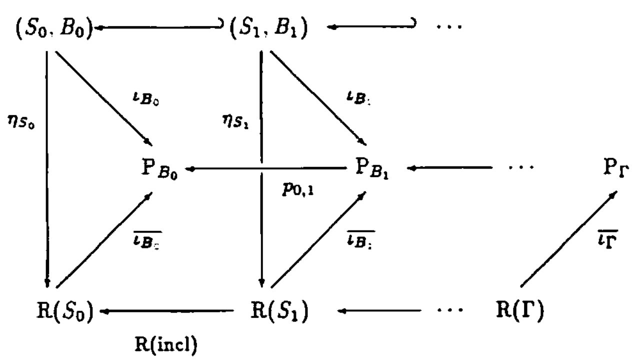
(b) *If $(S_1, B_1) \leq (S_0, B_0)$ in $\mathcal{S}$, then there exists a surjective morphism $p_{0,1}: \mathrm{P}_{B_1}(C,G) \to \mathrm{P}_{B_0}(C,G)$ such that the family of all open embeddings $\iota_B: S \to \mathrm{P}_B(C,G)$ with $(S,B) \in \mathcal{S}$ is a natural transformation between the inverse systems $\mathcal{S}$ and $(\mathrm{P}_B(C,G), p_{j,k})_{j \leq k}$.*

(c) *There exist the inverse limits of topological semigroups*

$$\begin{aligned} \mathrm{P}_\Gamma(C,G) &= \varprojlim\{\mathrm{P}_B(C,G) \mid (\exists S)\ (S,B) \in \mathcal{S}\} \quad \textit{and} \\ \mathrm{R}(\Gamma) &= \varprojlim\{\mathrm{R}(S) \mid (\exists B)\ (S,B) \in \mathcal{S}\} \end{aligned}$$

with limit maps $p_B: \mathrm{P}_\Gamma(C,G) \to (S,B)$ and $r_B: \mathrm{R}(\Gamma) \to (S,B)$. Moreover, there exists a morphism $\overline{\iota_\Gamma}: \mathrm{R}(\Gamma) \to \mathrm{P}_\Gamma(C,G)$ such that $p_B \circ \overline{\iota_\Gamma} = \overline{\iota_B} \circ r_B$ holds for all $(S,B) \in \mathcal{S}$.

The subsequent diagram illustrate the assertions of (3.3). For the sake of brevity we write P_B instead of $\mathrm{P}_B(C,G)$ for all $(S,B) \in \mathcal{S}$ and P_Γ instead of $\mathrm{P}_\Gamma(C,G)$.



The local semigroups (S,B) occuring in (3.3)(a) are given by upper sets with respect to the conal order (cf. the remarks after (3.1)). The most critical part of the proof is to establish the open embeddings $\iota_B: S \to \mathrm{P}_B(C,G)$ in part (a) of the preceding proposition. The rest follows by standard arguments involving limits [5, 23.3].

This result enables us to assign global topological semigroups to each pointed cone (C, L): Let $\tilde{G}$ be a simply connected Lie group with $L = \mathrm{L}(\tilde{G})$ and choose a germ Γ according to (3.3)(a). The topological semigroups

$$\mathrm{P}(C, L) = \mathrm{P}_{\Gamma}(C, \tilde{G}) \quad \text{and} \quad \mathrm{R}(C, L) = \mathrm{R}(\Gamma)$$

are called the *limit conal path semigroup* and the *limit relatively free semigroup* of the cone (C, L), respectively.

The germ Γ is determined by the conal order on suitable 1-neighborhoods in G and the choice of the representative inverse system $\mathcal{S} \in \Gamma$ obviously does not have any influence onto the construction of the above limits. It is not hard to show that the local structure of the simply connected Lie group completely determines the structure of the limit conal path semigroup [24, 4.4]. In other words, the choice of $\tilde{G}$ and Γ in the preceding definition is not essential.

Theorem 3.4. [24, 4.6] *Let $f:(C_1, L_1) \to (C_2, L_2)$ be a morphism of pointed cones, G_j simply connected Lie groups with $\mathrm{L}(G_j) = L_j$ $(j = 1, 2)$ and $\overline{f}: G_1 \to G_2$ the induced homomorphism with $f = \mathrm{L}(\overline{f})$.*

(a) *There exist germs $[S_j]$ of local Lie semigroups in G_j $(j = 1, 2)$ and a morphism $[\eta_f]: [S_1] \to [S_2]$ with $f = \mathrm{L}'([\eta_f]): \mathrm{L}'([S_1]) \to \mathrm{L}'([S_2])$.*

(b) *There exists a continuous homomorphism $f^{\mathcal{N}}: \mathcal{N}(C_1, G_1) \to \mathcal{N}(C_2, G_2)$, $\quad \alpha \mapsto \overline{f} \circ \alpha$.*

(c) *For each $(S_2, B_2) \in \mathcal{S}_2$ there exists $(S_1, B_1) \in \mathcal{S}_1$ such that the restriction of $\overline{f}$ to B_1 induces a morphism $f^r: (S_1, B_1) \to (S_2, B_2)$ of local semigroups. Moreover, there exists a continuous homomorphism $f^P_{1,2}: \mathrm{P}_{B_1}(C_1, G_1) \to \mathrm{P}_{B_2}(C_2, G_2)$ with $f^P_{1,2} \circ \kappa_{B_1} = \kappa_{B_2} \circ f^{\mathcal{N}}$.*

(d) *The assignments of objects $(C, L) \mapsto \mathrm{P}(C, L)$ or $\mathrm{R}(C, L)$ induce functors* P, R: **LCone** $\to$ **TSgrp** *such that the following diagrams commute:*

$$\begin{array}{ccccccc}
G_1 & \xleftarrow{\ \mathrm{end}\ } & \mathcal{N}(C_1, G_1) & \xrightarrow{\ \kappa_{B_1}\ } & \mathrm{P}_{B_1}(C_1, G_1) & \xleftarrow{\ p_{B_1}\ } & \mathrm{P}(C_1, L_1) \\
\downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f^{\mathcal{N}}} & & \downarrow{\scriptstyle f^P_{1,2}} & & \downarrow{\scriptstyle \mathrm{P}(f)} \\
G_2 & \xleftarrow[\ \mathrm{end}\ ]{} & \mathcal{N}(C_2, G_2) & \xrightarrow[\ \kappa_{B_2}\ ]{} & \mathrm{P}_{B_2}(C_2, G_2) & \xleftarrow[\ p_{B_2}\ ]{} & \mathrm{P}(C_2, L_2)
\end{array}$$

$$\begin{array}{ccccccc}
G_1 & \xleftarrow{\ \mathrm{incl}\ } & (S_1, B_1) & \xrightarrow{\ \eta_{S_1}\ } & \mathrm{R}(S_1) & \xleftarrow{\ r_{B_1}\ } & \mathrm{R}(C_1, L_1) \\
\downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f^r} & & \downarrow{\scriptstyle \mathrm{R}(f^r)} & & \downarrow{\scriptstyle \mathrm{R}(f)} \\
G_2 & \xleftarrow[\ \mathrm{incl}\ ]{} & (S_2, B_2) & \xrightarrow[\ \eta_{S_2}\ ]{} & \mathrm{R}(S_2) & \xleftarrow[\ r_{B_2}\ ]{} & \mathrm{R}(C_2, L_2)
\end{array}$$

The proof of this theorem rests on the detailed knowledge of the semigroups $\mathcal{N}(C_j, G_j)$ and the congruences κ_{B_j}, $(j = 1, 2)$. Some examples of conal path semigroups have been investigated in [23], [24].

It is remarkable that the simply connected group $\tilde{G}$ used in the definition of $\mathrm{P}(C, L)$ and $\mathrm{R}(C, L)$ can itself be obtained as the identity-component of (G, τ'), where G is an inverse limit and τ' is a locally connected refinement of the limit topology (cf. the observations at the beginning of this section). The members G_B of the corresponding inverse system are completely determined by the products defined in the local group B. Similarly, the conal path semigroups P_B are determined by the concatenation of conal curves α, β such that all of α, β and $\alpha\flat\beta$ stay in B. This analogy motivates the following constructions: If X is a topological monoid, choose the set of arc-components of open subsets of X as a base for a topology σ' on X. Moreover, let X^a be the arc-component of the identity in (X, σ'). Some straightforward verifications show, that X and (X, σ') have the same arcs and therefore also the same arc-components. The inclusion $X^a \hookrightarrow X$ is the pointwise co-unit of a coreflection into the category **AcLacSgrp** of arcwise connected and locally arcwise connected semigroups. Applied to the inverse limits $\mathrm{P}(C, L)$ and $\mathrm{R}(C, L)$ this procedure yields objects $\mathrm{P}^a(C, L)$ and $\mathrm{R}^a(C, L)$ in **AcLacSgrp**, and the assignments $(C, L) \mapsto \mathrm{P}^a(C, L)$ or $\mathrm{R}^a(C, L)$ are also functorial. Let p_B^a and r_B^a denote the corresponding restriction of the limit maps p_B and r_B, respectively. In contrast to the group case it is not known whether the maps maps r_B^a are coverings. It is conjectural that some suitable local semigroup (S, B) with prescribed tangent cone (C, L) can be openly embedded into $\mathrm{R}^a(C, L)$ and even $\mathrm{P}^a(C, L)$.

Acknowledgements

I gratefully acknowledge the valuable hints of J. Hilgert, K. H. Hofmann, J. D. Lawson and K.-H. Neeb. I also wish to express my gratitude to the Konrad-Adenauer-Stiftung for the scholarship it rewarded me.

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