Category Theory at Work
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Fubini's Theorem from a Categorical Viewpoint

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Abstract: Integrals can be defined by a universal measure. Then Fubini's theorem follows from the preservation of a binary coproduct by a left adjoint functor. Here we give only a simplified version, which works only for simple functions but shows the main ideas.

Introduction

The map χ which assigns to every element of a set algebra B its *characteristic function* is a *charge* (i.e. a finitely additive set function) on B with values in the vector space LB of *simple functions.* As F. Linton ([4],[5]) observed, this charge is even *universal*, i.e. every (vector-valued) charge factors uniquely over χ. The induced map assigns to each function its integral (see 3.2 below). Moreover, LB is even an *algebra* under pointwise multiplication, and χ is *multiplicative* in the sense that $\chi(1) = 1$ and $\chi(X \cap Y) = \chi(X) \cdot \chi(Y)$ for all $X, Y \in B$. It is easy to see that χ is even a universal multiplicative charge. Thus, for an algebra A, multiplicative charges $B \longrightarrow A$ correspond to algebra homomorphisms $LB \longrightarrow A$, and this bijection is natural in both arguments. On the other hand, multiplicative charges also correspond to Boolean homomorphisms $B \longrightarrow JA$, where JA is the *Boolean algebra of idempotents* of A (see 5.1 below). Thus, considering arbitrary Boolean algebras rather than set algebras (cf. [3]), we obtain an adjunction $L \dashv J$. Hence L preserves colimits, in particular binary coproducts. Since simple functions of *two variables* correspond to simple functions on a *coproduct* of two Boolean algebras, we obtain a version of *Fubini's theorem* (Section 6).

This only works for simple functions. By convergence arguments we can get a generalization to bounded measurable functions, if B is even a σ-algebra of sets. This can be done by considering Boolean algebras and vector spaces with sequential topologies subject to some conditions. But this is much more complicated, see [1]. On the other hand, the finitely additive version already gives a flavour of the "algebraic" part of Fubini's theorem. Moreover, since we are not concerned about convergence, we have no difficulties to develop the theory over an arbitrary field, even over a commutative unital ring.

So we always work over a *fixed commutative unital ring* R. All *tensor products* are taken over R, all R-algebras are *associative, commutative*, and *unital*, and all R-algebra homomorphisms *preserve the unit element.*

This work is an original contribution and will not appear elsewhere.

Likewise - but different from Linton's ([4]. [5]) notations - every Boolean algebra has a *top element* 1; the *bottom element* is denoted by 0. For a set algebra B on a set Ω we always assume $\Omega \in B$; the other axioms are $(X, Y \in B \Longrightarrow X \cap Y \in B)$ and $(X \in B \Longrightarrow \Omega \backslash X \in B)$. Then we also have $\emptyset \in B$ and $(X, Y \in B \Longrightarrow X \cup Y \in B)$. We refer to [2] and [6] for the theory of Boolean algebras.

1. Simple Functions

Let B be a set algebra on a set Ω. A *B-simple* function is a map $f : \Omega \longrightarrow R$ such that its image $f[\Omega] \subset R$ is finite and $f^{-1}[\{\zeta\}] \in B$ for all $\zeta \in R$. We denote the set of all B-simple functions by LB. For any $f, g \in LB$ and all $\alpha \in R$ the *pointwise sum* $f + g$ and the *pointwise multiple* αf are B-simple, and LB is an R-module under these operations.

For $X \in B$ we have the *characteristic function* $\chi(X) : \Omega \longrightarrow R$, where

$$\chi(X)(\omega) := \begin{cases} 1, & \text{if } \omega \in X, \\ 0, & \text{if } \omega \notin X. \end{cases}$$

One easily sees that $\chi(X) \in LB$ for all $X \in B$. Moreover, let FB be the free R-module over B and let $\eta : B \longrightarrow FB$ be the universal map. Then the universal property of FB renders a unique R-linear map $q : FB \longrightarrow LB$ with $q \circ \eta = \chi$.

1.1 Theorem: *For any set-algebra B, there is an exact sequence*

$$0 \longrightarrow K \hookrightarrow FB \xrightarrow{q} LB \longrightarrow 0,$$

where the submodule $K < FB$ is generated by all $\eta(X) + \eta(Y) - \eta(X \cup Y)$ with $X, Y \in B$, $X \cap Y = \emptyset$.

Proof: For K as in the theorem, we first prove the following

<u>Claim</u>: For every $s \in FB$, there exist $n \in \mathbb{N} \cup \{0\}$, $\alpha_1, \ldots, \alpha_n \in R$, $X_1, \ldots, X_n \in B$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $s - \sum_{i=1}^{n} \alpha_i \eta(X_i) \in K$.

Every $t \in FB$ has a representation $t = \sum_{i=1}^{k} \rho_i \eta(Z_i)$, where $k \in \mathbb{N} \cup \{0\}$, $\rho_i \in R$, $Z_i \in B$. We proceed by induction on k. The case $k = 0$ is trivial. Now assume that the claim is true for a given k and let $s = \sum_{i=1}^{k+1} \rho_i \eta(Z_i)$ be given. By induction hypothesis, for $t := \sum_{i=1}^{k} \rho_i \eta(Z_i)$ there exist $m \in \mathbb{N}$, $\beta_1, \ldots, \beta_m \in R$, $Y_1, \ldots, Y_m \in B$ with $Y_i \cap Y_j$ for $i \neq j$ and $t - \sum_{i=1}^{m} \beta_i \eta(Y_i) \in K$.
For every $i \leq m$ we have $Y_i \cap Z_{k+1} \in B$, $Y_i \backslash Z_{k+1} \in B$, $(Y_i \cap Z_{k+1}) \cap (Y_i \backslash Z_{k+1}) = \emptyset$, $(Y_i \cap Z_{k+1}) \cup (Y_i \backslash Z_{k+1}) = Y_i$, hence $\eta(Y_i \cap Z_{k+1}) + \eta(Y_i \backslash Z_{k+1}) - \eta(Y_i) \in K$.
From this we conclude $t - \sum_{i=1}^{m} \beta_i \eta(Y_i \cap Z_{k+1}) - \sum_{i=1}^{m} \beta_i \eta(Y_i \backslash Z_{k+1}) = \sum_{i=1}^{m} \beta_i (\eta(Y_i \cap Z_{k+1})$ $+ \eta(Y_i \backslash Z_{k+1})) = (t - \sum_{i=1}^{m} \beta_i \eta(Y_i)) - \sum_{i=1}^{m} \beta_i (\eta(Y_i \cap Z_{k+1}) + \eta(Y_i \backslash Z_{k+1}) - \eta(Y_i)) \in K$.

Now we have $Z_{k+1} = \bigcup_{i=1}^{m} (Y_i \cap Z_{k+1}) \cup Z'$ for $Z' := Z_{k+1} \backslash \bigcup_{i=1}^{m} Y_i$. By an easy induction argument we see that $\sum_{i=1}^{n} \eta(Y_i \cap Z_{k+1}) + \eta(Z') - \eta(Z_{k+1}) \in K$. Since $s = t + \rho_{k+1} \eta(Z_{k+1})$, we obtain

$s - \sum_{i=1}^{m}(\beta_i + \rho_{k+1})\eta(Y_i \cap Z_{k+1}) - \sum_{i=1}^{m}\beta_i\eta(Y_i \backslash Z_{k+1}) - \rho_{k+1}\eta(Z')$

$= t + \rho_{k+1}\eta(Z_{k+1}) - \sum_{i=1}^{m}\beta_i\eta(Y_i \cap Z_{k+1}) - \rho_{k+1}\sum_{i=1}^{m}\eta(Y_i \cap Z_{k+1}) - \sum_{i=1}^{m}\beta_i\eta(Y_i \backslash Z_{k+1}) - \rho_{k+1}(Z')$

$= (t - \sum_{i=1}^{m}\beta_i\eta(Y_i \cap Z_{k+1}) - \sum_{i=1}^{m}\beta_i\eta(Y_i \backslash Z_{k+1})) - \rho_{k+1}\sum_{i=1}^{m}(\eta(Y_i \cap Z_{k+1}) + \eta(Z') - \eta(Z_{k+1})) \in K.$

But the $2m+1$ sets $Y_i \cap Z_{k+1}, Y_i \backslash Z_{k+1}, Z'$ are pairwise disjoint. This finishes the induction and proves the claim.

Now we prove $\ker q = K$. On the one hand, for $X, Y \in \Omega$ with $X \cap Y = \emptyset$ we have $\chi(X \cup Y) = \chi(X) + \chi(Y)$ hence $q(\eta(X) + \eta(Y) - \eta(X \cup Y)) = q \circ \eta(X) + q \circ \eta(Y) - q \circ \eta(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cup Y) = 0$, proving $K \subset \ker q$.

Conversely, assume $s \in \ker q$. Then by the above claim there exist $n \in \mathbb{N} \cup \{0\}$, $\alpha_1, \ldots, \alpha_n \in R$, $X_1, \ldots, X_n \in B$ with $X_i \cap X_j = \emptyset$ for $i \neq j$ and $s - \sum_{i=1}^{n}\alpha_i\eta(X_i) \in K \subset \ker q$, hence $\sum_{i=1}^{n}\alpha_i\eta(X_i) = s - (s - \sum_{i=1}^{n}\alpha_i\eta(X_i)) \in \ker q$, thus $0 = q(\sum_{i=1}^{n}\alpha_i\eta(X_i)) = \sum_{i=1}^{n}\alpha_i(q \circ \eta)(X_i) = \sum_{i=1}^{n}\alpha_i\chi(X_i)$. If $\omega \in X_i$, then we have $\chi(X_{i_0})(\omega) = 1$, but for $i \neq i_0$ we have $X_i \cap X_{i_0} = \emptyset$, hence $\omega \notin X_i$ and $\chi(X_i)(\omega) = 0$. This gives $\alpha_{i_0} = \sum_{i=1}^{n}\alpha_i\chi_i(X_i)(\omega_i) = 0$, proving that for any i_0 either $X_{i_0} = \emptyset$ or $\alpha_{i_0} = 0$, hence $\alpha_{i_0}\eta(X_i) = 0$ or $\alpha_{i_0}\eta(X_i) = \alpha_{i_0}\eta(\emptyset)$.

This gives $\sum_{i=1}^{n}\alpha_i\eta(X_i) = \alpha\eta(\emptyset) \in K$ for some $\alpha \in R$, because $\eta(\emptyset) = \eta(\emptyset) + \eta(\emptyset) - \eta(\emptyset \cup \emptyset) \in K$. Now we obtain $s = (s - \sum_{i=1}^{n}\alpha_i\eta(X_i)) + \sum_{i=1}^{n}\alpha_i\eta(X_i) \in K$, proving $\ker q = K$, i.e. the above sequence is exact at FB.

Since the inclusion $K \hookrightarrow FB$ is trivially injective, the sequence is also exact at K. Moreover, if $f \in LB$, then $f[\Omega]$ is finite, e.g. $f[\Omega] = \{\alpha_1, \ldots, \alpha_n\}$. For every $i \in \mathbb{N}$, we have $X_i := f_i^{-1}[\{\alpha_i\}] \in B$, and it follows easily that $f = \sum_{i=1}^{n}\alpha_i\chi(X_i) = \sum_{i=1}^{n}\alpha_i(q \circ \eta)(X_i)$ $= q(\sum_{i=1}^{n}\alpha_i\eta(X_i)) \in q[FB]$. This proves that q is surjective, i.e. the sequence is also exact at LB.
□

2. Functions of Two Variables

Let B_i be a set algebra on Ω_i $(i \in \{0, 1\})$. Then for $\Omega := \Omega_0 \times \Omega_1$ and the canonical projections $p_i : \Omega \longrightarrow \Omega_i$ $(i \in \{0, 1\})$ we consider the smallest set algebra B on Ω containing all $X_0 \times X_1 = (X_0 \times \Omega) \cap (\Omega_0 \times X_1) = p_0^{-1}[X_0] \cap p_1^{-1}[X_1]$, where $X_0 \in B_0, X_1 \in B_1$. Then $v_i : B_i \longrightarrow B, X \longmapsto p_i^{-1}[X]$ is a Boolean homomorphism, and B is generated by $v_0[B_0] \cup v_1[B_1]$ as a set algebra, i.e. as a Boolean algebra.

2.1 Theorem: *In the above situation, B is a coproduct of B_0 and B_1 in the category* Bool *of Boolean algebras, with injections $v_i : B_i \longrightarrow B$ $(i \in \{0, 1\})$.*

Proof: The result follows immediately from the following coproduct criterion:

2.2 Lemma: *Let B, B_0, B_1 be Boolean algebras and let $v_i : B_i \longrightarrow B$ be Boolean homomorphisms. Then B is a coproduct of B_0 and B_1 with injections $v_i : B_i \longrightarrow B$ $(i \in \{0, 1\})$, if and only if the following conditions hold:*

(i)*B is generated by $v_0[B_0] \cup v_1[B_1]$ as a Boolean algebra.*
(ii) *If $x_i \in B_i \backslash \{0\}$ $(i \in \{0,1\})$, then $v_0(x_0) \wedge v_1(x_1) \neq 0$ in B.*
Proof: "$\Rightarrow$" The construction of the coproduct (=sum) $B := B_0 \amalg B_1$ is standard (cf. e.g. [2]). Let $u_i : B_i \longrightarrow B$ be the coproduct injections and let $B' \subset B$ be the Boolean subalgebra generated by $u_0[B_0] \cup u_1[B_1]$. Then the u_i restrict to Boolean homomorphisms $u_i' : B_i \longrightarrow B'$ $(i \in \{0,1\})$. For the inclusion $j : B' \hookrightarrow B$ we have $j \circ u_i' = u_i$ $(i \in \{0,1\})$. The coproduct property renders a unique $r : B \longrightarrow B'$ with $r \circ u_i = u_i'$ for $i \in \{0,1\}$. This leads to $j \circ r \circ u_i = j \circ u_i' = u_i$ for $i \in \{0,1\}$. Since $j \circ r$ and the identity $B \longrightarrow B$ are both Boolean homomorphisms, they coincide by the uniqueness condition of the coproduct property. In particular, j is surjective, hence $B' = B$, proving (i).

For $x_0 \in B_0 \backslash \{0\}$, $x_1 \in B_1 \backslash \{0\}$ there are Boolean homomorphisms $h_i : B_i \longrightarrow 2$ with $h_i(x_i) = 1$, where $2 = \{0,1\}$ is the two-element Boolean algebra. Then the coproduct property yields a unique homomorphism $h : B \longrightarrow 2$ with $h \circ u_i = h_i$ for $i \in \{0,1\}$. This gives $h(u_0(x_o) \wedge u_1(x_1)) = (h \circ u_0)(x_0) \wedge (h \circ u_1)(x_1) = h_0(x_0) \wedge h_1(x_1) = 1 \wedge 1 = 1 \neq 0 = h(0)$, hence $u_0(x_0) \wedge u_1(x_1) \neq 0$, proving (ii).

"$\Leftarrow$". Let $u_i : B_i \longrightarrow B_0 \amalg B_1$ be the coproduct injections $(i \in \{0,1\})$. Then the coproduct property gives a unique Boolean homomorphism $v : B_0 \amalg B_1 \longrightarrow B$ with $v \circ u_i = v_i$ $(i \in \{0,1\})$. Now v is surjective by (i). Moreover, every $z \in B_0 \amalg B_1$ admits a representation $z = \bigvee_{k=1}^{n} (u_0(x_{0k}) \wedge u_1(x_{1k}))$, where $x_{ik} \in B_i$ $(i \in \{0,1\}, k = 1, \ldots, n)$ (cf. [2]). This can also be seen directly in the following way: the set of all z such that both z and z^c admit such representations is closed under the Boolean operations and contains the images of the injections; hence it is all of $B_0 \amalg B_1$.

If we have $z \neq 0$, then we get $u_0(x_{0k_0}) \wedge u_1(x_{1k_0}) \neq 0$ for some k_0, hence $u_i(x_{ik_0}) \neq 0$, thus $x_{ik_0} \neq 0$ for both $i \in \{0,1\}$. Now (ii) gives $0 \neq v_0(x_{0k_0}) \wedge v_1(x_{1k_0}) \leq \bigvee_{k=1}^{n} (v_0(x_{0k}) \wedge v_1(x_{1k}))$ $= \bigvee_{k=1}^{n} ((v \circ u_0(x_{0k})) \wedge (v \circ u_1(x_{1k}))) = v(\bigvee_{i=1}^{n} (u_0(x_{0k}) \wedge u_1(x_{1k}))) = v(z)$, hence $v(z) \neq 0$. But $B_0 \amalg B_1$ is an abelian group under the symmetric difference (i.e. $z + w := (z \wedge w^c) \vee (z^c \wedge w) = (z \vee w) \wedge (z \wedge w)^c$, where ^{c} denotes complementation), and v is a group homomorphism. Since we have seen
$\ker v = \{0\}$, we see that v is also injective.

Thus v is bijective and hence a Boolean isomorphism, because Bool is monadic over Set. Then the result follows immediately. □

3. Universal Charges

Now we work over an *abstract Boolean algebra* B. A map μ from B to an R-module A is called a *charge*, if $\mu(x \vee y) = \mu(x) + \mu(y)$ holds for all $x, y \in B$ with $x \wedge y = 0$. Then we also have $\mu(0) = \mu(0 \vee 0) = \mu(0) + \mu(0)$, hence $\mu(0) = 0$.

Now we consider the universal map $\eta : B \longrightarrow FB$ (where FB is the free R-module over B). Let $K < FB$ be the submodule generated by all elements $\eta(x) + \eta(y) - \eta(x \vee y)$, where $x, y \in B$ with $x \wedge y = 0$. We define $LB := FB/K$, and for the canonical projection $q : FB \longrightarrow LB$ we define $\chi := q \circ \eta : B \longrightarrow LB$.

3.1 Theorem: *$\chi : B \longrightarrow LB$ is a charge. For any R-module A and any charge $\mu : B \longrightarrow A$, there exists a unique R-linear map $l : B \longrightarrow A$ with $l \circ \chi = \mu$.*

Proof: For $x, y \in B$ with $x \wedge y = 0$ we have $\chi(x) + \chi(y) - \chi(x \vee y) = q(\eta(x) + \eta(y) - \eta(x \vee y)) = 0$, because $\eta(x) + \eta(y) - \eta(x \vee y) \in K$. This proves that χ is a charge.

If $\mu : B \longrightarrow A$ is a charge, the universal property of FB yields a unique R-linear map $h : FB \longrightarrow A$ with $h \circ \eta = \mu$. For $x, y \in B$ with $x \wedge y = 0$ we obtain $h(\eta(x) + \eta(y) - \eta(x \vee y)) = \mu(x) + \mu(y) - \mu(x \vee y) = 0$. This gives $K \subset \ker h$, and thus there is a unique R-linear map $l : LB \longrightarrow A$ with $l \circ q = h$. This l satisfies $l \circ \chi = l \circ q \circ \eta = h \circ \eta = \mu$, and clearly l is uniquely determined by the equation $l \circ \chi = \mu$. □

3.2 Remarks: For $R = \mathbb{R}$ and B a set algebra, we see from 1.1 that LB and χB here are the same as in section 1 (up to a canonical isomorphism). Thus we may assume that LB is the space of simple functions and that for any $X \in B$, $\chi(X)$ is the characteristic function. Then for μ, l as in 3.1 we have $l \circ \chi(X) = \mu(X) = \int \chi(X) d\mu$. But l and the integral are both R-linear, and since $\chi[B]$ generates LB as a vector space, we see that $l(f) = \int f d\mu$ for all $f \in LB$.

But the above definition of LB does not depend on a representation of B as a set algebra. So the elements of LB can be viewed as generalized simple functions. Moreover, for $f \in LB$ and μ, l as above we introduce the notation $\int f d\mu := l(f)$.

By $M(B, A)$ we denote the set of all charges on B with values in A. Then $M(B, A)$ is even an R-module under pointwise operations, and $M : \mathbf{Bool}^{op} \times \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R$ becomes a functor in the obvious way. For any fixed B, A the map $\mathbf{Mod}_R(LB, A) \longrightarrow M(B, A), l \longmapsto l \circ \chi$ is clearly an R-module isomorphism. This even gives a natural isomorphism between the functors $M(B, \square), \mathbf{Mod}_R(LB, \square) : \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R$, i.e. $M(B, \square)$ is represented by LB. For any Boolean homomorphism, the induced natural transformation $M(h, \square) : M(B', \square) \longrightarrow M(B, \square)$ corresponds to a unique R-linear map $\bar{L}h : LB \longrightarrow LB$ by the Yoneda lemma. It follows easily that $\bar{L} : \mathbf{Bool} \longrightarrow \mathbf{Mod}_R$ is a functor. Thus we obtain:

3.3 Corollary: *The functors $M : \mathbf{Bool}^{op} \times \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R$ and $(B, A) \longmapsto \mathbf{Mod}_R(LB, A)$ are naturally isomorphic.* □

4. Multiplicative Charges

Till now we have not used the fact that the simple functions over a set algebra are closed under *pointwise multiplication*; they even form an (associative, commutative and unital) *R-algebra*. We can generalize this multiplication to our "pointless" approach in the following way: let B be a Boolean algebra. Then "$\wedge$": $B \times B \longrightarrow B$ is an associative and commutative binary operation on B with unit element 1. Since $(\eta(x))_{x \in B}$ is a base of FB, there exists a unique bilinear multiplication "$\cdot$" on FB such that $\eta(x) \cdot \eta(y) = \eta(x \wedge y)$ for all $x, y \in B$. Since associativity and commutativity are multilinear identities, we see that FB is associative and commutative. By the same reason, $1 := \eta(1)$ is a unit element of FB.

4.1 Lemma: *Let B be a Boolean algebra and let $K < FB$ be the submodule generated by all $\eta(x) + \eta(y) - \eta(x \vee y)$, where $x, y \in B$ with $x \wedge y = 0$. Then K is an ideal in FB.*

Proof: For $x, y, z \in B$, we have $(\eta(x) + \eta(y) - \eta(x \vee y)) \cdot \eta(z) = (\eta(x) \cdot \eta(z) + \eta(y) \cdot \eta(z) - \eta(x \vee y) \cdot \eta(z)) = \eta(x \wedge z) + \eta(y \wedge z) - \eta((x \vee y) \wedge z) = \eta(x \wedge z) + \eta(y \wedge z) - \eta((x \wedge z) \vee (y \wedge z)) \in K$, because $(x \wedge z) \wedge (y \wedge z) = x \wedge y \wedge z = 0 \wedge z = 0$. Since the $\eta(z)$ generate FB as an R-module and by definition of K, 4.1 follows immediately. $\square$

Now the quotient $LB := FB/K$ is also an R-algebra. For the projection $q : FB \longrightarrow LB$ and for $\chi := q \circ \eta$ as in 3.1 we see that $\chi(1) = q \circ \eta(1) = q(1) =: 1$ is a unit element of LB, and for $x, y \in B$ we have $\chi(x \wedge y) = q\circ\eta(x \wedge y) = q(\eta(x) \cdot \eta(y)) = (q\circ\eta(x)) \cdot ((q\circ\eta(y)) = \chi(x) \cdot \chi(y)$. If A is an R-algebra, we call a charge $\mu \in M(B, A)$ *multiplicative*, if $\mu(1) = 1$ and $\mu(x \wedge y) = \mu(x) \cdot \mu(y)$ for all $x, y \in B$.

4.2 Theorem: *Let B be a Boolean algebra. Then $\chi : B \longrightarrow LB$ is a multiplicative charge. For every R-algebra A and every multiplicative charge $\mu : B \longrightarrow A$ the map $LB \longrightarrow A, f \longmapsto \int f d\mu$ is a (unital) R-algebra homomorphism.*

Proof: It has already been shown that χ is multiplicative. Integration is R-linear by 3.1. Moreover, we obtain $\int 1 d\mu = \int \chi(1) d\mu = \mu(1) = 1$. For $x, y \in B$ we have $\int(\chi(x) \cdot \chi(y)) d\mu$ $= \int \chi(x \wedge y) d\mu = \mu(x \wedge y) = \mu(x) \cdot \mu(y) = \int \chi(x) d\mu \cdot \int \chi(y) d\mu$. Since the $\chi(x)$ generate LB as an R-module, by bilinearity we obtain $\int(f \cdot g) d\mu = \int f d\mu \cdot \int g d\mu$ for all $f, g \in LB$. $\square$

4.2. means that the universal charge is also a *universal multiplicative charge*. Conversely, for a multiplicative charge $\mu : B \longrightarrow A$ and an R-algebra homomorphism $l : A \longrightarrow A'$ it follows immediately that $l \circ \mu : B \longrightarrow A'$ is also a multiplicative charge. By $M^\times(B, A)$ we denote the set of A-valued multiplicative charges. Then the above maps $M^\times(B, A) \longrightarrow \mathrm{Alg}_R(LB, A)$, $\mu \longmapsto \int \square d\mu$ and $\mathrm{Alg}_R(LB, A) \longrightarrow M^\times(B, A)$, $l \longmapsto l \circ \chi$ are inverse to each other.

Similarly mutliplicative charges compose with Boolean homomorphisms on the right, hence $M^\times : \mathrm{Bool}^{op} \times \mathrm{Alg}_R \longrightarrow \mathrm{Set}$ is a functor. Note that $M^\times(B, A)$ is not a submodule of $M(B, A)$; it is neither closed under sums nor under scalar multiples. For a fixed B the above bijection gives a natural transformation between the functors $M^\times(B, \square)$ and $\mathrm{Alg}_K(LB, \square)$.

From the Yoneda lemma it follows that $L : \mathrm{Bool} \longrightarrow \mathrm{Alg}_R$ is a functor. This can also been seen directly from 4.2: If $h : B \longrightarrow B'$ is a Boolean homomorphism and if $\chi : B \longrightarrow LB$, $\chi' : B' \longrightarrow LB'$ are the universal charges, then $\chi' \circ h \in M^\times(B, LB')$, and thus the map $Lh : LB \longrightarrow LB'$,
$f \longmapsto \int f d(\chi' \circ h)$ is an R-algebra homomorphism. The functorial properties of L follow from standard argument. From this we obtain:

4.3 Corollary: *The functor $M^\times$ is naturally isomorphic to the functor*
$\mathrm{Bool}^{op} \times \mathrm{Alg}_R \longrightarrow \mathrm{Set}, \quad (B, A) \longmapsto \mathrm{Alg}_R(LB, A)$. $\square$

5. Boolean Algebras of Idempotents

We have seen that the functor $M^\times$ is "representable in the second variable". Now we shall give a "representation" of $M^\times$ "in the first variable". For an (associative, commutative and unital) R-algebra A let JA be the set of all *idempotents*, i.e. all $f \in A$ with $f^2 = f$. Then, by straightforward arguments, JB is a Boolean algebra with top element 1 and bottom element 0, where $f \wedge g := fg$, $f^c := 1 - f$ and $f \vee g := f + g - fg$ for all $f, g \in JB$.

5.1 Theorem: *For every R-algebra A, the inclusion $\iota : JA \longrightarrow A$ is a multiplicative charge. For every Boolean algebra and every multiplicative charge $\mu : B \longrightarrow A$ there is a unique Boolean homomorphism $h : B \longrightarrow JA$ with $\iota \circ h = \mu$.*

Proof: Straightforward; just note that $\mu(x) = \mu(x \wedge x) = \mu(x)^2$, hence $\mu(x) \in JA$ for every $\mu \in M^\times(B, A)$, $x \in B$.

5.2 Remark: If $2 := 1 + 1$ is not a zero divisor in A, (i.e. if the additive group of A has no 2-torsion), we have some kind of converse of the above statement: every $\mu \in M(B, A)$ with $\mu(1) = 1$ and $\mu[B] \subset JA$ is multiplicative. Indeed for $x, y \in B$ with $x \wedge y = 0$ we obtain $\mu(x) + \mu(y) = \mu(x \vee y) = \mu(x \vee y)^2 = (\mu(x) + \mu(y))^2 = \mu(x)^2 + 2\mu(x)\mu(y) + \mu(y)^2 = \mu(x) + \mu(y) + 2\mu(x)\mu(y)$, hence $2\mu(x)\mu(y) = 0$ and then $\mu(x)\mu(y) = 0$ by our hypothesis. Now for every $x, y \in B$ we obtain $\mu(x) \cdot \mu(y) = (\mu(x \wedge y) + \mu(x \wedge y^c)) \cdot (\mu(x \wedge y) + \mu(x^c \wedge y)) = \mu(x\wedge y)^2 + \mu(x\wedge y)\cdot\mu(x^c\wedge y) + \mu(x\wedge y^c)\cdot\mu(x\wedge y) + \mu(x^c\wedge y)\cdot\mu(x\wedge y^c) = \mu(x\wedge y)+0+0+0 = \mu(x\wedge y)$. If 2 is invertible in R (in particular, if R is a field of characteristic $\neq 2$), then 2 is invertible and hence a non-zero-divisor in every R-algebra.

5.1 leads to the desired "representation"of $M^\times$ "in the first argument". Note that $J : \mathrm{Alg}_R \longrightarrow \mathbf{Bool}$ is trivially a functor.

5.3 Corollary: $M^\times : \mathbf{Bool}^{op} \times \mathrm{Alg}_R \longrightarrow \mathbf{Set}$ *is naturally isomorphic to the functor* $(B, A) \longmapsto \mathbf{Bool}(B, JA)$. □

Composing the natural isomorphisms of 4.3 and 5.3 we obtain bijections $\mathrm{Alg}_R(LB, A) \cong M^\times(B, A) \cong \mathbf{Bool}(B, JA)$, natural in both arguments. Now we immediately obtain the following:

5.4 Theorem: *The functor* $J : \mathrm{Alg}_R \longrightarrow \mathbf{Bool}$ *is right adjoint to* $L : \mathbf{Bool} \longrightarrow \mathrm{Alg}_R$. □

5.5 Remark: If B is a set algebra on a set Ω, then by 1.1 we may view the elements of LB as simple functions. We easily see that $f \in LB$ is idempotent if and only if $f(\omega)$ is idempotent in R for all $\omega \in \Omega$. In particular, if R is a connected ring (i.e. $R \neq \{0\}$ and $0, 1$ are the only idempotents of R), then an idempotentent simple function f attains only the values 0 and 1, hence $f = \chi(X)$ for some unique $X \in B$. This means that $\bar{\chi} : B \longrightarrow JLB,\ X \longmapsto \chi(X)$ is a bijection and hence an isomorphism of Boolean algebras. But $\bar{\chi}$ is the unit of the adjunction from 5.4 at B.

Since every Boolean algebra is isomorphic to a set algebra by Stone's representation theorem (cf. [2],[6]), we see that the unit of the above adjunction is always an isomorphism. This can also been seen by a direct translation of the above argument. Thus J is full and faithful in this case. But note that 5.4 still holds for arbitrary R.

In the classical situation $A = R = \mathbb{R}$, by 5.1 multiplicative charges on a Boolean algebra B correspond to Boolean homomorphisms $B \longrightarrow J\mathbb{R} \cong 2$, i.e. to ultrafilters (or maximal ideals) in B.

6. Fubini's Theorem

From 5.4 we see that the left-adjoint functor L preserves colimits, in particular binary coproducts. But in Alg_R, the coproduct is the tensor product, because all algebras are assumed to be commutative and unital. For R-algebras A_0, A_1, the multiplication on $A_0 \otimes A_1$ is the unique bilinear operation with $(f_0 \otimes f_1)(g_0 \otimes g_1) := f_0 g_0 \otimes f_1 g_1$ for all $f_i, g_i \in A_i$, where $i \in \{0, 1\}$. The coproduct injections $j_i : A_i \longrightarrow A_0 \otimes A_1$ are given by $j_0(f_0) := f_0 \otimes 1,\ \ j_1(f_1) := 1 \otimes f_1$. If A is an R-algebra and if $l_i : A_i \longrightarrow A$ are R-algebra homomorphisms, then the unique R-algebra homomorphisms $l : A_0 \otimes A_1 \longrightarrow A$ with $l \circ j_i = l_i$ $(i \in \{0, 1\})$ is the unique R-linear map with $l(f_0 \otimes f_1) = l_0(f_0) \cdot l_1(f_1)$ for all $f_0 \in A_0,\ f_1 \in A_1$.

6.1 Theorem: *Let* B_0, B_1 *be Boolean algebras and let* $B_i \longrightarrow B_0 \amalg B_1$ *be the coproduct injections.*

Then the unique R-linear map $k : LB_0 \otimes LB_1 \longrightarrow L(B_0 \amalg B_1)$ with $k(f_0 \otimes f_1) := Lu_0(f_0) \cdot Lu_1(f_1)$ is an R-algebra isomorphism.

Proof: By the above argument, k is the unique R-algebra homomorphism induced by the coproduct property of $LB_0 \otimes LB_1$. Since the coproduct of B_0 and B_1 is preserved by L, k is an isomorphism. □

Now let $\chi_i : B_i \longrightarrow LB_i$ $(i \in I)$ and $\chi : B_0 \amalg B_1 \longrightarrow L(B_0 \amalg B_1)$ be the universal charges. For arbitrary (not necessary multiplicative) charges $\mu_i \in M(B_i, R)$, we define $l_i : LB_i \longrightarrow R$ by $l_i(f) := \int f d\mu$ $(i \in \{0,1\},\ f \in LB_i)$. For the multiplication isomorphism $m : R \otimes R \longrightarrow R$, $\alpha \otimes \beta \longmapsto \alpha\beta$, we have $\mu := m \circ (l_0 \otimes l_1) \circ k^{-1} \circ \chi \in M(B, R)$. We call μ the product charge of μ_0 and μ_1. For $f \in L(B_0 \amalg B_1)$ we get $\int f d\mu = (m \circ (l_0 \otimes l_1) \circ k^{-1})(f)$.

For $x_0 \in B_0,\ x_1 \in B_1$, we have $k(\chi_0(x_0) \otimes \chi_1(x_1)) = (Lu_0 \circ \chi_0(x_0)) \cdot (Lu_1 \circ \chi_1(x_1))$ $= (\chi \circ u_0(x_0)) \cdot (\chi \circ u_1(x_1)) = \chi(u_0(x_0) \wedge u_1(x_1))$, because χ is multiplicative. This leads to $\mu(u_0(x_0) \wedge u_1(x_1)) = (m \circ (l_0 \circ l_1) \circ k^{-1})(u_0(x_0) \wedge u_1(x_1)) = (m \circ (l_0 \otimes l_1))(\chi_0(x_0) \otimes \chi_1(x_1))$ $= m((l_0 \circ \chi_0)(x_0) \otimes (l_1 \circ \chi_1)(x_1)) = m(\mu_0(x_0) \otimes \mu_1(x_1))) = \mu_0(x) \cdot \mu_1(x_1)$.

Now return to the case that B_i is a set algebra on Ω_i for $i \in \{0,1\}$. Then by 2.1 we may regard $B_0 \amalg B_1$ as a set algebra on $\Omega_0 \times \Omega_1$, and for $X_0 \in B_0,\ X_1 \in B_1$ we have $X_0 \times X_1 = u_0(X_0) \cap u_1(X_1)$, hence $\mu(X_0 \times X_1) \cong \mu_0(X_0) \cdot \mu_1(X_1)$. Then μ is the product charge in the usual sense, i.e. it is defined like the "product measure without σ-additivity".

Now consider the elements of LB_i $(i \in \{0,1\})$ and $L(B_0 \amalg B_1)$ as simple functions. Then for any $\omega_0 \in \Omega_0,\ \omega_1 \in \Omega_1$ we have $\chi(X_0 \times X_1)(\omega_0, \omega_1) = \chi_0(X_0)(\omega_0) \cdot \chi_1(X_1)(\omega_1)$, hence
$\int \chi(X_0 \times X_1)(\omega_0, \omega_1)\mu(d(\omega_0, \omega_1)) = \mu(X_0 \times X_1) = \mu_0(X_0) \cdot \mu_1(X_1)$
$= \int \chi_0(X_0)(\omega_0)\mu_0(d\omega_0) \cdot \int \chi(X_1)(\omega_1)\mu_1(d\omega_1) = \int(\chi_0(X_0)(\omega_0) \cdot \int \chi_1(X_1)(\omega_1)\mu_1(d\omega_1))\mu_0(d\omega_0)$
$= \int(\int \chi_0(X_0)(\omega_0) \cdot \chi_1(X_1)(\omega_1)\mu_1(d\omega_1))\mu_0(d\omega_0) = \int(\int \chi(X_0 \times X_1)(\omega_0, \omega_1)\mu_1(d\omega_1))\mu_0(d\omega_0)$.

Now every $X \in B_0 \amalg B_1$ is a disjoint union $\bigcup_{\nu=1}^{n} (X_0^{(\nu)} \times X_1^{(\nu)})$, where $X_i^{(\nu)} \in B_i$ $(i \in \{0,1\},\ \nu \in \{1,\dots,n\})$, thus $\chi(X) = \sum_{\nu=1}^{n} \chi(X_0^{(\nu)} \times X_1^{(\nu)})$. Every $f \in L(B_0 \amalg B_1)$ can be written as $f = \sum_{\rho=1}^{r} \alpha_\rho \chi(X_\rho)$, where $\alpha_1, \dots, \alpha_r$ are the (finitely many) values of f and $X_\rho := f^{-1}[\{\alpha_\rho\}]$ $(\rho \in \{1,\dots,r\})$. So we get the finitely additive Fubini's theorem for simple functions:

6.2 Corollary: *Let B_i be a set algebra on Ω_i, let $\mu_i \in M(B, R)$ and let μ be the product measure of μ_0 and μ_1. Then $\int f(\omega_0, \omega_1)\mu d(\omega_0, \omega_1)) = \int(\int f(\omega_0, \omega_1)\mu_1(d\omega_1))\mu_0(d\omega_0)$ holds for all $f \in L(B_o \amalg B_1)$.*

Proof: The identity is linear in f. But by the above argument, f is a linear combination of functions of the form $\chi(X)$, $X \in B_0 \amalg B_1$. But each such X is a sum of functions of the form $\chi(X_0 \times X_1)$, $X_0 \in B_0$, $X_1 \in B_1$. For functions of this type the identity has been shown above. Thus it holds for all $f \in L(B_0 \amalg B_1)$. □

7. Final Remarks

The proof of 6.2 is just translating 6.1 from the "pointless" approach to "classical" measure theory. If one wants to work in the "pointless" situation, one may consider 6.1 as a version of Fubini's theorem. Our approach at least makes it clear, why product charges exist; it just follows from coproduct preservation by a left-adjoint functor.

This adjunction in 5.4 may also be interesting in itself, but the only thing we need about the functor J is that it is right-adjoint to L. Multiplicative charges were introduced solely for this purpose; 5.4 and 6.1 do not refer to multiplicative charges any more. Though $\mathbb{R}$-valued multiplicative charges are quite trivial (see 5.5), the functorial behaviour of algebra valued multiplicative charges is the key to our version of Fubini's theorem.

For our functorial machinery we have to work with charges with values in modules, even if we are only interested in integration of R-valued simple functions with respect to R-valued charges. This is in contrast to the classical theory, where the vector-valued Bochner-integral seems more complicated than the scalar-valued Lebesgue-integral; vector-valued measures look even more difficult.

F. Linton ([4],[5]) already mentions the universal property of χ and the natural isomorphism of 5.3, but he does not consider multiplicative charges. His approach to Fubini's theorem for abstract σ-algebras (i.e. Boolean algebras with countable joins) is quite different. As a "pointless" substitute for measurarable real-valued functions one might consider σ-algebra homomorphisms from the σ-algebra of Borel subsets of $\mathbb{R}$ to a given σ-algebra. This is quite similar to Linton's approach, which involves some more technical difficulties, because Linton does not require Boolean algebras to have top elements.

In our approach, top elements are convenient, because they correspond to unit elements of R-algebras. The latter are crucial, because the tensor product is not a coproduct in the category of not necessary unital R-algebras. Since the tensor product of unital not necessarily commutative R-algebras is not a coproduct either, commutativity is also needed here. It is also needed in order to get a Boolean algebra structure on JA; in a non-commutative R-algebra products of idempotents need not be idempotent. For instance, the product $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ of the idempotent real matrices $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not idempotent.

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