Category Theory at Work
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Convexity Theories Ʊ – Back to the Future

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ABSTRACT: In this note we present an outline of what might be called *algebraic convexity theory*. We describe briefly the salient features of this theory, state few conjectures, and indicate promising directions for future research.

The definition of convexity theories requires the presence of a normed unital semi-ring R. Unital semi-rings are described by exactly the same axioms as unital rings, except that the existence of additive inverses is not assumed. Normed unital semi-rings are unital semi-rings R equipped with a norm $\|\Box\| : R \to \mathbf{R}$ that is subject to the standard requirements of a norm: in particular $\|1\| = 1$. A slight modification of the notion of Cauchy sequence in a normed ring leads here to the concept of Cauchy sequence in a normed unital semi-ring. Similarly one has the notion of limit of a Cauchy sequence in a normed unital semi-ring. If every Cauchy sequence in a normed unital semi-ring has a unique limit then we speak of a Banach semi-ring.

Let $R^{\mathbb{N}}$ denote the set of sequences $\alpha_\bullet := (\alpha_1, \alpha_2, \ldots)$ with entries in the normed semi-ring R. The support $\mathrm{supp}(\alpha_\bullet)$ is the set $\{i \in \mathbb{N} : \alpha_i \neq 0\}$ and the cardinality of $\mathrm{supp}(\alpha_\bullet)$ is called the length $lg(\alpha_\bullet)$. A subset Γ of $R^{\mathbb{N}}$ is said to be finitary, if $\alpha_\bullet \in \Gamma$ implies $lg(\alpha_\bullet) < \aleph_0$; otherwise Γ is said to be infinitary. Given a subset Γ of $R^{\mathbb{N}}$, we denote the elements of $\Gamma^{\mathbb{N}}$ by $\beta_\bullet^\Box$, with the understanding that $\beta_\bullet^j$ is in Γ, for all $j \in \mathbb{N}$. Define $\langle \alpha_\Box, \beta_\bullet^\Box \rangle \in R^{\mathbb{N}}$, for $\alpha_\Box \in \Gamma$ and $\beta_\bullet^\Box \in \Gamma^{\mathbb{N}}$, by

$$\langle \alpha_\Box, \beta_\bullet^\Box \rangle_i := \sum \{\alpha_j \beta_i^j : i \in \mathbb{N}\} .$$

If Γ is finitary then $\langle \alpha_\Box, \beta_\bullet^\Box \rangle$ is well defined under the following additional assumptions:

(i) R is a Banach semi-ring.

(ii) $\|\alpha_\bullet\| \leq 1$ and $\|\beta_\bullet^j\| \leq 1$, $j \in \mathbb{N}$, where $\|\alpha_\bullet\| := \sum\{\|\alpha_i\| : i \in \mathbb{N}\}$ (and $\|\beta_\bullet^j\|$ correspondingly).

Therefore, whenever we are dealing with an infinitary $\Gamma \subseteq R^{\mathbb{N}}$, we assume - without stating so explicitly - that R is a Banach semi-ring.

DEFINITION: Let R be a normed semi-ring. A (left) *convexity theory* Γ over R is a subset of $R^{\mathbb{N}}$ such that

($\Gamma 0$) $\|\alpha_\bullet\| \leq 1$ for all $\alpha_\bullet \in \Gamma$,

($\Gamma 1$) $\delta_\bullet^j := (\delta_1^j, \delta_2^j, \ldots) \in \Gamma$ for all $j \in \mathbb{N}$, where δ_i^j is the usual Kronecker symbol.

($\Gamma 2$) $\langle \alpha_\Box, \beta_\bullet^\Box \rangle \in \Gamma$, for all $\alpha_\bullet, \beta_\bullet^1, \beta_\bullet^2, \ldots \in \Gamma$.

A number of useful formulae involving the composition $\langle \Box, \Box \rangle$ are known.

DEFINITION: Let Γ' and Γ be convexity theories (over R' resp. R). Then a map $\varphi : \Gamma' \longrightarrow \Gamma$ is called a *morphism of convexity theories if*

This work is an original contribution and will not appear elsewhere.

(M1) $\varphi(\delta^j_\bullet) = \delta^j_\bullet$ for all $j \in \mathbb{N}$,

(M2) $\varphi(\langle \alpha_\Box, \beta^\Box_\bullet \rangle) = \langle \varphi(\alpha_\Box), \varphi^{\mathbb{N}}(\beta^\Box_\bullet) \rangle$, for all $\alpha_\bullet, \beta^1_\bullet, \beta^2_\bullet, \ldots \in \Gamma'$, where $\varphi^{\mathbb{N}} : \Gamma'^{\mathbb{N}} \longrightarrow \Gamma^{\mathbb{N}}$ is the induced map.

These two definitions give rise to the category **Conv** of convexity theories. There is a functor S from the category **Conv** to the category **Mon** of monoids. It assigns to each convexity theory Γ over R the submonoid $S(\Gamma)$ of $R^\times$ - the multiplicative monoid underlying R - consisting of precisely those $\rho \in R^\times$ for which $(\rho, 0, 0, \ldots) \in \Gamma$ holds; $S(\Gamma)$ is called the *monoid of scalars of* Γ. **Conv** plays a role somewhat similar to the category of unital rings and more information on **Conv** would be quite useful in our theory. One item that would be particularly interesting is a, at least partial, classification of the convexity theories over a given normed semi-ring. Indeed, in the case $R = \mathbb{R}$ all convexity theories containing the standard convexity theory $\Omega_C := \{\alpha_\bullet \in \mathbb{R}^{\mathbb{N}} : \|\alpha_\bullet\| = 1, lg(\alpha_\bullet) < \aleph_0, \alpha_i \geq 0$ for all $i \in \mathbb{N}\}$ can be characterized ([8]), the take-off point for this classification is Kuhn's Theorem ([2]).

Let Γ be a convexity theory and let X be a set. An operation of Γ on X is a map $\Gamma \times X^{\mathbb{N}} \ni (\alpha_\bullet, x^\bullet) \mapsto \langle \alpha_\bullet, x^\bullet \rangle \in X$. X together with such an operation is called a Γ-algebra. Following [5] we have the following two definitions

DEFINITION: Let X be a Γ-algebra. Then X is called Γ-*convex module*, if

($\Gamma C1$) $\langle \delta^j_\bullet, x^\bullet \rangle = x^j$ for all $j \in \mathbb{N}, x^\bullet \in X^{\mathbb{N}}$.

($\Gamma C2$) $\langle \langle \alpha_\Box, \beta^\Box_\bullet \rangle, x^\bullet \rangle = \langle \alpha_\Box, \langle \beta^\Box_\bullet, x^\bullet \rangle \rangle$ for all $\alpha_\bullet \in \Gamma, \beta^\Box_\bullet \in \Gamma^{\mathbb{N}}, x^\bullet \in X^{\mathbb{N}}$, where $\langle \beta^\Box_\bullet, x^\bullet \rangle^j := \langle \beta^j_\bullet, x^\bullet \rangle$.

DEFINITION: Let $\varphi : \Gamma' \to \Gamma$ be a morphism of convexity theories. Then a map f from the Γ'-convex module X' to the Γ-convex module X is called a *morphism of convex modules* over φ, if

$$f(\langle \alpha_\bullet, x^\bullet \rangle) = \langle \varphi(\alpha_\bullet), f^{\mathbb{N}}(x^\bullet) \rangle \quad \text{for all} \quad \alpha_\bullet \in \Gamma', x^\bullet \in X'^{\mathbb{N}},$$

where $f^{\mathbb{N}} : X'^{\mathbb{N}} \to X^{\mathbb{N}}$ is the induced map. If $\varphi = id_\Gamma$, then we speak of a *morphism of* Γ-*convex modules.*

If Γ is fixed, then the totality of Γ-convex modules together with their morphisms – composed set-theoretically – forms the category ΓC of Γ-*convex modules.* ΓC is an algebraic category. As such there are free Γ-convex modules on any set. There is an explicit construction of these free Γ- convex modules, that resembles the construction of the l_1-spaces on a given set. $S(\Gamma)$ equipped with the obvious Γ-convex structure is the free Γ-convex module on the set $\{1\}$; Γ itself is free on the set $\{\delta^j_\bullet : j \in \mathbb{N}\}$. Various computational rules for Γ-convex modules can be proved by verifying them for these free objects. Each Γ-convex module carries an intrinsic semi-norm $\|\Box\|_\Gamma$ whose general properties resemble those of a norm; in particular one has $\|x\|_\Gamma \leq 1$, for all $x \in X$. It can be used to define various special kinds of Γ-convex modules. (For special cases see [6] and [9])

Given the notion of normed (Banach) semi-ring R there is an obvious concept of normed (Banach) left semi-module M. A morphism $f : M' \longrightarrow M$ between two left semi-modules over the same normed semi-ring is said to be contractive, if $\|f(m')\| \leq \|m'\|$, for all $m' \in M'$. The category of normed (Banach) left semi-modules over R with their contractive morphisms is denoted by $\mathbf{R\text{-}Mod_1}$. Note that a contractive morphism $f : M' \longrightarrow M$ maps the closed unit ball $O(M')$ of M' to the closed unit ball $O(M)$ of M. Moreover, if Γ is a convexity theory over R, the closed unit ball $O(M)$ carries a canonical Γ-structure making it a Γ-convex module $\hat{O}_\Gamma(M)$. Since the contractive morphism $f : M' \longrightarrow M$, when restricted to the closed unit balls, is a morphism of Γ-convex modules $\hat{O}_\Gamma(f) : \hat{O}_\Gamma(M') \longrightarrow \hat{O}_\Gamma(M)$, we obtain a functor $\hat{O}_\Gamma$: R-Mod$_1$ $\longrightarrow$ ΓC. $\hat{O}_\Gamma$ has a left adjoint M_Γ : ΓC $\longrightarrow$ R-Mod$_1$. This left adjoint was constructed in a special case in [5], where it was denoted by S. M_Γ can be used to study the interior of Γ-convex modules.

It is of considerable importance for the investigation of ΓC to develop a theory of quotients of Γ-convex modules. This, of course, amounts to investigating congruence relations on Γ-convex modules. The effect of a congruence relation on the interior of a Γ-convex module X is fairly easy to describe, but the effect on the boundary is much more complex. However, the two most important cases can be handled reasonably well; that is the case, where $X = S(\Gamma)$ or X = the free Γ-convex module on the two-element set. The quotient modules of $S(\Gamma)$, i.e. the cyclic Γ-convex modules, define the *type* of an element x of a Γ-convex module X as follows: if $f : S(\Gamma) \longrightarrow X$ is the unique morphism with $f(1) = x$, then $im(f)$ is called the type of x in X. It would be very interesting to know more about the way in which various types are distributed over a given Γ-convex module. Additionally, similar to ordinary module theory, these types play a significant role in finding cogenerators for the category ΓC. In [5] and [3] the types of two particular convexity theories are explicitly determined. To find all quotient modules of the free Γ-convex module on the two-element set is important, as it reveals the structure of the submodule X' of X, that is generated by two arbitrary elements. This program was carried out for a particular convexity theory in [7]. In the category ΓC the notion of torsion theory can be formulated following [1]. However, a slight modification is needed, as a convexity theory need not contain the zero sequence $O_\bullet$. The semi-norm allows the construction of torsion theories in ΓC, that have no analogue in ordinary module theory. As is the case in ordinary module theory, the torsion theories in ΓC correspond to certain topologies in Γ.

Let $\varphi : \Gamma' \longrightarrow \Gamma$ be a morphism of convexity theories. Then φ gives rise to the functor $V(\varphi)$: ΓC $\longrightarrow$ Γ'C, called the *restriction-of-scalars functor* ; it assigns to each $X \in \Gamma$C the same set X on which Γ' operates by the formula $\langle\alpha'_\bullet, x^\bullet\rangle := \langle\varphi(\alpha'_\bullet), x^\bullet\rangle$, and to each morphism $f : X \longrightarrow Y$ the same map, which turns out to be a morphism in Γ'C. One can easily see that $V(\varphi)$ has a left adjoint $C(\varphi)$: Γ'C $\longrightarrow$ ΓC, and an explicit construction of the *completion functor*, or *extension-of-scalars functor*, $C(\varphi)$ is readily available. Of particular interest is the unit of the adjunction $\varepsilon : \Gamma'\mathrm{C} \longrightarrow V(\varphi)C(\varphi)$. In case X' is regular, that is $\|x'\|_\Gamma = 0$ implies $x' = O_X$, $\varepsilon_{X'}$ is conjectured to be an injective map. More generally, the congruence relation on X' included by $\varepsilon_{X'}$, is of considerable interest and should be studied in detail. In the same vein, one should be able to characterize among all objects of ΓC those of form $C(\varphi)(X')$, for same $X' \in \Gamma'$C.

A theory of dimension of Γ-convex modules can be developed by defining the dimension of X as the isomorphism class of $M_\Gamma(X)$. In a sense, the dimension of X captures the part of X that is separated. A related yet quite distinct issue is that of a *minimal system of generators* of X. To some degree, minimal systems of generators in conjunction with the dimensioin are a measure of the "roundness" of the boundary $bdy(X) := \{x \in X : \|x\|_\Gamma = 1\}$ of Γ-convex modules. Under certain compactness assumptions one obtains very general Krein-Milman type theorems for Γ-convex modules X. They remain in force under passage from X to that Γ-convex quotient module of X, in which the interior $\mathrm{int}(X) := \{x \in X : \|x\|_\Gamma < 1\}$ is one congruence class, while all other congruence classes consist of a single element only. More generally, the study of the behavior of minimal systems of generators under quotient morphisms yields interesting phenomena.

Various versions ot the Hahn-Banach Theorem have been proved in certain special classes: for total convexity [4], for positive convexity [10], and for general positive convexity theories [8]. It is likely that these results can be extended to some if not all convexity theories over arbitrary Banach algebras. The main problem here is, whether a convexity theory over a, say real, Banach algebra A has a large enough intersection with the convexity theory $\Omega_{\mathbb{R}} := \{\alpha_\bullet \in \mathbb{R}^{\mathbb{N}} : \|\alpha_\bullet\| \leq 1\}$; such reality problems would become much more tractable by corresponding advances in the classification of convexity theories over

A. The classical Hahn-Banach Theorem, which calls for morphisms to $O_\Gamma(R)$, or Γ-convex submodules of $O_\Gamma(R)$, can be extended to furnish morphisms to products of various types, an issue that is closely related to determining cogenerators for the category ΓC. At the same time, these Hahn-Banach type theorems lead to en extensive duality theory for Γ-convex modules and to a theory of dual Γ-convex modules.

The category ΓC fails to be an additive category. Still there is the possibility of launching homological investigations involving Γ-convex modules. However, the appropriate concept to use here is that of $\Lambda - \Theta$ -convex bimodules, where Λ is a left convexity theory, Θ is a right convexity theory (in the sense that ($\Gamma 2$) is replaced by "$\langle \beta^{\square}_{\bullet}, \alpha_{\square} \rangle \in \Theta$"), and an additional axiom ($\Lambda - \Theta$ $C3$), stating the obvious associativity condition for $\Lambda - \Theta$- convex bimodules, is added to $(\Lambda C1) \equiv (\Theta C1) \equiv (\Gamma C1)$, and $(\Lambda C2) \equiv (\Gamma C2)$ and the corresponding $(\Theta C2)$. Then $\Lambda C(X', X)$ is a right Θ-convex module, and for appropriate bimodules X and X'' the tensor product $X' \otimes_\Theta X''$ can be defined as usual (by shifting scalars form the right of X' to the left of X''); a special case is discussed in [5]. Hence, in the situation described, we have the basic constructs for homological algebra. In case all convexity theories involved are finitary, there is a broader concept, that is amenable to homological investigations. It is that of *linear theory*. A (left) linear theory over a (not necessarily normed) semi-ring R is a subset Λ of $R^{\mathbb{N}}$ such that

$$
\begin{aligned}
&(\Lambda 0) \qquad lg(\alpha_\bullet) < \aleph_0 , \qquad \text{for all } \alpha_\bullet \in \Lambda, \\
&(\Lambda 1) \quad \equiv \quad (\Gamma 1), \\
&(\Lambda 2) \quad \equiv \quad (\Gamma 2),
\end{aligned}
$$

hold. In case Λ is the set of all $\alpha_\bullet \in R^{\mathbb{N}}$ with $lg(\alpha_\bullet) < \aleph_0$, the Λ-linear modules are precisely the R-modules, while for other choices of Λ various generalizations of the R-module concept appear (see e.g.: W. Bos and G. Wolff, Mitt. Math. Sem. Giessen 129, 1–15, 1978; F. Ostermann and J. Schmidt, J. Reine a. Angew. Math. 224, 44–57, 1966).

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