Category Theory at Work
H. Herrlich, H.-E. Porst (eds.)
© Heldermann Verlag Berlin 1991
277-288

Free Structures

J. Reiterman, V. Trnková

Abstract

We consider some situations where free structures (mostly free algebras) play a role; we discuss their use, their existence and the methods how to construct them. It has not been our aim that the present paper should exhaust the topic and also our references are only samples of existing literature.

I. Classical universal algebra

The importance of free algebraic structures was recognized in classical universal algebra [14],[9]. The latter deals with algebras of a given type Ω (which is a *set* of operation symbols with given finite arities). An *Ω-algebra* A is a couple $A = (|A|, (\sigma_A)_{\sigma\in\Omega})$ where $|A|$ is the ***underlying set*** and the σ_A's are *operations:* if $\sigma \in \Omega$ has arity m then $\sigma_A : |A|^m \longrightarrow |A|$. A ***homomorphism*** $f : A \longrightarrow B$ of Ω-algebras is a map $f : |A| \longrightarrow |B|$ commuting with operations in the sense that

$$f\sigma_A(a_1, \dots, a_m) = \sigma_B(fa_1, \dots, fa_m)$$

for every $\sigma \in \Omega$ and every $a_1, \dots, a_m \in |A|$ where m is the arity of σ. All Ω-algebras and their homomorphisms form a category which will be denoted by **Ω-alg**.

I.1. Absolutely free algebras

We are going to introduce the concept of an absolutely free Ω-algebra over a set X of generators. Intuitively, this is an algebra the underlying set of which contains X and the operations of which are defined as free as possible. It is constructed as follows.

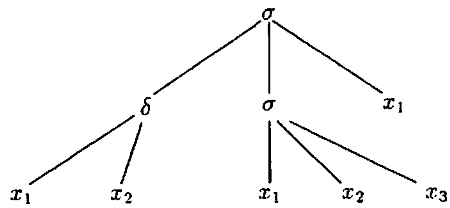
Given a set T, denote by $\Omega(T)$ the set of all formal expressions $\sigma(t_1, \dots, t_m)$ where $\sigma \in \Omega$ is of arity m and $t_1, \dots t_m \in T$. Define a chain of sets X_i by induction:

$X_0 = X$;
if X_i is defined put $X_{i+1} = X \cup \Omega(X_i)$; finally put
$|X^\bullet| = \bigcup_{i=0}^{\infty} X_i$
and define operations in $|X^\bullet|$ by
$\sigma_{X^\bullet}(t_1, \dots t_m) = \sigma(t_1, \dots, t_m)$.

This work is an original contribution and will not appear elsewhere.

The algebra X^* obtained is called the *absolutely Ω-free algebra over X*. Its elements are called *Ω-terms with variables in X.*

Elements of X^* can be visualized as finite trees whose leaves are labeled by elements of X and other vertices are labeled by operation symbols. E.g., for Ω consisting of one ternary operation symbol σ and one binary operation symbol δ, the tree



represents the term $\sigma(\delta(a_1, x_2), \sigma(x_1, x_2, x_3),\ x_1)$. Observe that *every term involves a finite number of variables.*

Theorem. The absolutely free algebra X^* has the following universal property: For every Ω-algebra A and every map $f : X \longrightarrow |A|$ there is a unique homomorphism $f^* : X^* \longrightarrow A$ with $f^*\nu = f$ where $\nu : X \longrightarrow |X^*|$ is the inclusion.

Proof. We have to construct a homomorphism $f^* : X^* \longrightarrow A$ extending f. We proceed by induction. On X_0, let f^* coincide with f. If f^* has been defined on X_i and $t \in \Omega(X_i)$, $t = \sigma(t_1, \dots, t_m)$, put

$$f^*(t) = \sigma_A(f^*t_1, \dots, f^*t_m).$$

Note that this means that $f^*\sigma_{X^*}(t_1, \dots, t_m) = \sigma_A(f^*t_1, \dots, f^*t_m)$ which forces f^* to be a homomorphism. It is unique for X^* is obviously generated by X.

I.2. Equations.

Let us start with an example. A *groupoid* is an Ω-algebra A where Ω consists of a unique binary operation symbol $\sigma = \circ$. Let us write $x_1 \circ x_2$ instead of $\sigma(x_1, x_2)$. A is a *semigroup* if is satisfies the associativity equation $(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3)$. This means: if we replace the variables x_i by arbitrary particular elements of A and compute the expressions on both sides then the results are the same.

Observe that the expressions in question are elements of X^* where $X = \{x_1, x_2, x_3, \dots\}$ and the results of the computations are $f^*((x_1 \circ x_2) \circ x_3), f^*(x_1 \circ (x_2 \circ x_3))$ where f is the map assigning the particular elements of A to variables. This leads to the following general definition:

An equation for Ω-algebras is a couple $t = t'$ where $t, t' \in X^*$ (where X^* is the absolutely free Ω-algebra over the countable set $X = \{x_1, x_2, x_3, \dots\}$ of variables).

An Ω-algebra A is said to *satisfy* an equation $t = t'$ if $f^*t = f^*t'$ for every map $f : X \longrightarrow |A|$.

Let us turn our attention to *derived equations.* E.g., a semigroup A satisfies not only the basic associativity equation but many other equations: the trivial ones, such as $x_1 \circ x_2 = x_1 \circ x_2$, and those which are formally deduced from associativity such as $x_1 \circ (x_2 \circ (x_3 \circ x_4)) = (x_1 \circ x_2) \circ (x_3 \circ x_4)$.

Let us formulate general *deduction rules:*

(1) For every term $t \in X^\bullet, t = t$;

(2) From equations $t = u, u = v$, one can deduce $t = v$ and $u = t$;

(3) Let t, t' be terms in variables $x_1, \ldots, x_m$, let $t_1, \ldots, t_m, t'_1, \ldots, t'_m$ be arbitrary terms, and let s, s' be terms obtained from t, t', resp., by replacing each x_i by t_i in t and by replacing each x_i by t'_i in t'.
Then from $t = t', t_1 = t'_1, \ldots, t_m = t'_m$ one can deduce $s = s'$.

Note that s and s' above can be expressed as $f^\bullet t$ and $g^\bullet t'$, resp., where $f^\bullet : X^\bullet \longrightarrow X^\bullet$ is the unique homomorphic extension of any map $f : X \longrightarrow |X^\bullet|$ with $fx_i = t_i$ and analogously $g^\bullet$ is determined by $g : X \longrightarrow |X^\bullet|$ with $gx_i = t'_i$. Then (3) implies

(3a) From $t_1 = t'_1, \ldots, t_m = t'_m$ one can deduce
$\sigma_{X^\bullet}(t_1, \ldots, t_m) = \sigma_{X^\bullet}(t'_1, \ldots, t'_m)$ for every $\sigma \in \Omega$ of arity m.

(3b) For every $f : X \longrightarrow |X^\bullet|$, from $t = t'$, one can deduce $f^\bullet t = f^\bullet t'$.

I.3. Varieties

Denote by (Ω, E)-**alg** the full subcategory of Ω-**alg** consisting of all algebras satisfying a given set E of equations. Denote by $E^\bullet$ the set of all equations which can be deduced from E. Then $E^\bullet$ can be constructed by induction as follows:

Let E_o be the set of equations consisting of the trivial equations $t = t$ and of all equations from E.
If E_n has been defined, let E_{n+1} be the set of all equations which can be deduced from equations in E_n using the above deduction rules (1) - (3).
Finally put $E^\bullet = \bigcup_{n=0}^{\infty} E_n$.

Remark. Every algebra A in (Ω, E)-**alg** satisfies all $E^\bullet$-equations.

(Hint: Using the deductions rules, prove by induction on n that A satisfies all E_n-equations.)

Define a relation $\sim_E$ on $X^\bullet$ by $t \sim_E t'$ iff $t = t'$ is an $E^\bullet$-equation.

Proposition. The relation $\sim_E$ is a congruence on $X^\bullet$,[1] and the corresponding factor-algebra $X^\bullet/\sim_E$ [2] satisfies all E-equations, i.e., $X^\bullet/\sim_E$ belongs to (Ω, E)-**alg**.

Proof. $\sim_E$ is an equivalence relation by (1), (2) and it is a congruence by (3a).

Let $t = t'$ be an E-equation. Denote by $h : X^\bullet \longrightarrow X^\bullet/\sim_E$ the canonical map. Let $f : X \longrightarrow X^\bullet/\sim_E$ be any map. Choose $g : X \longrightarrow |X^\bullet|$ with $hg = f$. Then $hg^\bullet = f^\bullet$ because

[1] i.e., an equivalence on $|X^\bullet|$ satisfying $\sigma_{X^\bullet}$ $(t_u, \ldots, t_m) \sim_E \sigma_{X^\bullet}(t'_1, \ldots, t'_m)$ for all $\sigma \in \Omega$ of arity m and $t_i, t'_i \in |X^\bullet|$ with $t_i \sim_E t'_i$, $i = 1, \ldots, m$,

[2] with $|X^\bullet|/\sim_E$ as underlying set and operations $\tilde{\sigma}$ given by $\tilde{\sigma}([t_1], \ldots, [t_m]) := [\sigma_{X^\bullet}(t_1, \ldots, t_m)]$ for all $\sigma \in \Omega$ of arity m and equivalence classes $[t_1], \ldots, [t_m] \in X^\bullet/\sim_E$

both $hg^\bullet$ and $f^\bullet$ are homomorphisms from $X^\bullet$ to $X^\bullet/\sim_E$ and they coincide on X. As $g^\bullet t = g^\bullet t'$ is an $E^\bullet$-equation by (3b), one gets $g^\bullet t \sim_E g^\bullet t'$. Hence $hg^\bullet t = hg^\bullet t'$, i.e., $f^\bullet t = f^\bullet t'$.

Note that we have a natural map $\nu_E : X \longrightarrow |X^\bullet/\sim_E|$, viz the composition of the inclusion $\nu : X \longrightarrow |X^\bullet|$ and the canonical homomorphism $h : X^\bullet \longrightarrow X^\bullet/\sim_E$.

Theorem. The algebra $X^\bullet/\sim_E$ together with the natural map $\nu_E : X \longrightarrow |X^\bullet/\sim_E|$ has the following universal property: For every (Ω, E)-algebra A and every map $f : X \longrightarrow |A|$ there is a unique homomorphism $f_E^\bullet : X^\bullet/\sim_E \longrightarrow A$ with $f_E^\bullet \nu_E = f$.

Proof. $f : X \longrightarrow |A|$ extends to a homomorphism $f^\bullet : X^\bullet \longrightarrow A$; if $t \sim_E t'$, i.e., if $t = t'$ is an $E^\bullet$-equation then A satisfies $t = t'$ by the above Remark; hence $f^\bullet t = f^\bullet t'$. But then $f^\bullet$ factors through the canonical homomorphism $h : X^\bullet \longrightarrow X^\bullet/\sim_E$ in the sense that $f^\bullet = f_E^\bullet h$ for some homomorphism $f_E^\bullet$ as required.

According to the above universal property, we call $X^\bullet/\sim_E$ the *free* (Ω, E)*-algebra over* X.

I.4. Completeness theorem

Free algebras provide a useful tool in proofs. The applications of free algebras are based on the fact that they are objects where syntax and semantics of an algebraic theory meet. This is used to prove the important.

Completeness Theorem for Equational Logic: An equation for Ω-algebras holds in all (Ω, E)-algebras iff it can be deduced from E.

Proof. Every (Ω, E)-algebra satisfies all $E^\bullet$-equations by the above Remark. Conversely, let an equation $t = t'$ be satisfied in all (Ω, E)-algebras. Then it is satisfied in the free (Ω, E)-algebra $X^\bullet/\sim_E$ and for the canonical map $\nu_E : X \longrightarrow |X^\bullet/\sim_E|$ we have $\nu_E^\bullet t = \nu_E^\bullet t'$. But $\nu_E^\bullet$ is nothing else but the canonical map $h : X^\bullet \longrightarrow X^\bullet/\sim_E$ for both $\nu_E^\bullet$ and h are homomorphisms extending ν_E. Hence $ht = ht'$, i.e., $t \sim_E t'$, thus $t = t'$ is an $E^\bullet$-equation.

I.5. Syntax semantics relation-operations

The completeness theorem above establishes a canonical relation between syntax and semantics of classical algebraic theories in case of equations. The following result of Lawvere [19] essentially does the same for operations. Its proof is based on free algebras.

Given an (Ω, E)-algebra A, every Ω-term $t \in X^\bullet$ with variables $x_1, \dots, x_n$ *induces* an n-ary operation $t_A : |A|^n \longrightarrow |A|$, viz

$$t_A(a_1, \dots, a_n) = k_E^\bullet h(t)$$

where $k : X \longrightarrow |A|$ is any map with $k(x_i) = a_i$ $(i = 1, \dots, n)$, $k_E^\bullet : X^\bullet/\sim_E \longrightarrow A$ is its canonical extension and $h : X^\bullet \longrightarrow X^\bullet/\sim_E$ is the canonical map.

Observation. The operations t_A constitute a natural transformation t from $|-|^n$ to $|-|$ where $|-| : (\Omega, E)\text{-alg} \longrightarrow \mathbf{Set}$ is the underlying functor, i.e., the diagram

$$\begin{array}{ccc} |A|^n & \xrightarrow{\;t_A\;} & |A| \\ {\scriptstyle f^n}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ |B|^n & \xrightarrow{\;t_B\;} & |B| \end{array}$$

commutes for every homomorphism $f : A \longrightarrow B$ of (Ω, E)-algebras.

Proof. Let $a_1, \dots, a_n \in |A|$ and let $k : X \longrightarrow |A|$ be a map with $k(x_i) = a_i$ $(i = 1, \dots, n)$. Then $ft_A(a_1, \dots, a_n) = fk_E^\bullet h(t) = (fk)_E^\bullet h(t) = t_B(fa_1, \dots, fa_n)$.

Theorem. Every natural transformation $\tau : |-|^n \longrightarrow |-|$, where $|-| : (\Omega, E)\text{-alg} \longrightarrow \mathbf{Set}$ is the underlying functor, is induced by an Ω-term.

Proof. Denote by F the free (Ω, E)-algebra $X^\bullet / \sim_E$ and by $h : X^\bullet \longrightarrow F$ the canonical homomorphism. Choose $t \in X^\bullet$ with $ht = \tau_F(\nu_E(x_1), \dots, \nu_E(x_n))$. We claim that τ is induced by t. Indeed, let A be an (Ω, E)-algebra and $a_1, \dots, a_n \in |A|$. Consider any map $k : X \longrightarrow |A|$ with $k(x_i) = a_i$ $(i = 1, \dots, n)$ and the induced homomorphism $k_E^\bullet : X^\bullet \longrightarrow A$. Then $\tau_A(a_1, \dots, a_n) = \tau_A(k(x_1), \dots, k(x_n)) = \tau_A(k_E^\bullet \nu_E(x_1), \dots, k_E^\bullet \nu_E(x_n)) =$ $k_E^\bullet(\tau_F(\nu_E(x_1), \dots, \nu_E(x_n))) = k_E^\bullet h(t) = t_A(a_1, \dots, a_n)$.

II. Infinitary universal algebra

The first presentation of infinitary universal algebra is due to Slominski [29]. There appears that many results on finitary algebras generalize to the case that Ω is a set of operation symbols whose arities are arbitrary cardinal numbers rather than just finite ones. In particular, free algebras exist and can be used both to prove the completeness theorem for infinitary equational logic and to establish a canonical correspondence between terms and transformations. This remains true even if Ω is a *proper class* but, for every cardinal n, n-ary terms taken modulo E form a *set* only (see Linton [20] who generalized the above mentioned Lawvere's approach). Note that these proper class algebraic theories with the smallness property are called *varietal*. Examples of varietal theories are that of *compact Hausdorff spaces* and of *frames*.

For non-varietal theories free algebras need not exist; this is the case for the theory of complete lattices whose operations are joins and meets of arbitrary arities, and equations are infinitary analogues of the lattice theory laws; analogously for complete Boolean algebras. The non-existence of free complete lattices and free complete Boolean algebras was proved by Hales [15] and Gaifman [13]. Still the statement of the completeness theorem holds for both theories [15]. However, the correspondence between terms and transformations fails [27]: there are transformations which are not induced by terms: in fact, there is a proper class of such *wild* transformations. Thus the equational theory induced by the forgetful functor of the category of complete lattices is larger than the original theory of complete lattices. It is an open problem whether the category of algebras for the larger theory is equivalent to the category of complete lattices. Another open problem: do complete Boolean algebras admit wild transformations?

III: Many-sorted algebras: dynamic algebras

Many-sorted algebras are used in various applications of algebraic methods. These algebras have no more just one underlying set but a collection (X_s) of underlying sets where s runs through a set S of *sorts*. A classical example is given by vector spaces over fields: a vector space X over a field R can be regarded as a two-sorted algebra: its underlying sets, X and R, are equipped with operations defined on X and on R and with an *inter-sort operation* $\alpha \in R, x \in X \mapsto \alpha x \in X$.

Many-sorted algebras have been exploited in several branches of computer science, e.g., in algebraic specification [12]. We shall consider another example in detail: *dynamic algebras.*

Dynamic algebras were introduced by Kozen [17] and Pratt [26] to form models of propositional dynamic logic. Dynamic logic deals with propositions whose truth value varies. Consider, e.g., a command in a program, "**if** $x > 0$ **then** $x := -x$." Here "$x > 0$" is a proposition the validity of which can be changed by commands such as "$x := -x$". This can be formalized as follows.

We are given a set S of *states* of a computer and a set $\mathcal{B}$ of *propositions.* Each proposition can be identified with the set of states in which it holds. Then the usual operations with propositions, conjunction, disjunction and negation, coincide with the set-theoretical operations union, intersection, complement. Thus $\mathcal{B}$ is a Boolean algebra of subsets of S. Further, we are given a set $\mathcal{R}$ of (in general: non-determinsitic) *actions.* They enable the computer to move from one state to another and thus they can be described as binary relations on S. Operations with actions are *choice* $(a, b \longrightarrow a \cup b)$, *composition* $(a, b \longrightarrow ab)$ and *iteration* $(a \longrightarrow a^*)$. The influence of actions on propositions is expressed by means of an intersort operation $p \in \mathcal{B}, a \in \mathcal{R} \longrightarrow ap \in \mathcal{B}$ the meaning of which is "p can be true after the application of a". The resulting pair $(\mathcal{B}, \mathcal{R})$ with the above operations is called a *Kripke structure.* It can be viewed as a two-sorted algebra satifying the axioms

$$
\begin{array}{ll}
(\mathcal{B}\wedge, \vee, 0 \text{ is a Boolean algebra} & a0 = 0 \\
a(p \vee q) = ap \vee aq & (ab)p = a(bp) \\
p \vee aa^*p \leq a^*p \leq a^*(ap - p) & (a \vee b)p = ap \vee bp
\end{array}
$$

These are the axioms of dynamic algebras which form an equational variety of two-sorted algebras. As in the one-sorted case, free algebras in a variety exist. Their use can be demonstrated by an algebraic proof of the *completeness theorem for dynamic logic.* The theorem states that the tautologies of the propositional dynamic logic are precisely those which can be deduced from Segerberg's axioms [30]. Equivalently: a Boolean equation holds in all finite Kripke structures iff it can be deduced from the above axioms of dynamic algebras. The theorem is an immediate consequence of the following result of Pratt [26]: every free dynamic algebra can be embedded into a product of finite Kripke structures. Indeed, it follows that equations which are valid in finite Kripke structures hold in free dynamic algebras as well. And, as in the one-sorted case, a free algebra satisfies precisely those equations which can be deduced from the axioms of the theory.

The theory of dynamic algebras can be extended by adding a further operation, the *test* $? : \mathcal{B} \longrightarrow \mathcal{R}$, satisfying the axiom $(?p)q = p \wedge q$. Using this operation, constructs like "**if then** else" or "while do" are incorporated to dynamic logic. The resulting algebras - test algebras - are investigated in [28]. It is proved that a free test algebra can be embedded into a product of finite test structures, but the situation is more complicated. E.g., for dynamic algebras, one can confine oneself to *separable* algebras - those in which distinct actions have distinct effects ($a \neq b$ implies $ap \neq bp$ for some p). Surprisingly, free separated test algebras do not exist; free test algebras (unlike free dynamic algebras) are not separable [28].

IV. Algebras with more general carriers

In topological algebra, the carriers of the algebraic structure are topological spaces rather than just sets and the operations are continuous. This is the case for topological groups or topological vector spaces. Free algebras do exist in equational varieties of topological algebras. Sometimes these are just the ordinary free algebras equipped with a suitable topology but often no explicit description of that topology in known [23], [25].

Here we shall consider another type of algebras - those whose carriers are posets (= partially ordered sets). The idea of using ordered algebras in computer science originated with Dana Scott [10] who modeled flow diagrams as members of a complete lattice the operations of which preserve joins of directed sets. Markowski, Rosen and Reynolds used ordered algebras to investigate semantics of programming languages. The most general approach in this direction was proposed by the ADJ's [1] who considered a quite general concept of order-continuity:

A *subset system* is a rule Z asigning to each poset P a family $Z[P]$ of its subsets such that, for every order preserving map $f : P \longrightarrow Q, A \in Z[P]$ implies $f[A] \in Z[Q]$. A poset P is called Z-*complete* if every $A \in Z[P]$ has a join in P. An order preserving map of posets is $Z-$*continuous* if it preserves joins of Z-sets. A $Z-$*continuous algebra* (of some type Ω) is an algebra (of that type) whose carrier is a Z-complete poset and whose operations are Z-continuous.

Z-continuous algebras were used in such a way that their type represents the syntax and the initial Z-continuous algebras represents the semantics. The *initial algebra* is defined to be an algebra with just one homomorphism to any other algebra; thus it is nothing else but the free Z-continuous algebra over the initial Z-complete poset. The ADJ's constructed [1] the initial Z-continuous algebra in the special case of a finitary type and $Z = \omega$ (where $\omega[P]$ is the set of chains in P indexed by non-negative integers). Its members are trees labeled by operation symbols (as in classical universal algebra) but they have possibly infinite branches. A complete description of free Z-continuous algebras (for any type and any subset system Z) was given in [3]. Their elements are still more complicated labeled trees. However, the existence of free continuous algebras in equational varieties is an open problem. They do exist in the finitary case but the description can be very complicated, see, e.g., that for free continuous semilattices [5].

A different but similar type of algebras are separately Z-continuous algebras whose operations are required to be *separately Z-continuous* (i.e., Z-continuous in each of its variables). Free separately Z-continuous algebras need not exist [4].

V. Functor algebras

We are going to explain now one of the categorical approaches to algebras. Consider, e.g., a *groupoid*, i.e., an algebra X with one binary operation. Its operation is a map $\delta : X \times X \longrightarrow X$ with $(x, y) \mapsto x \cdot y$; equivalently, $\delta : FX \longrightarrow X$, where F is the functor from the category of sets into itself with

$$FX = X \times X, \quad Ff = f \times f.$$

A homomorphism from a groupoid X to a groupoid Y is a map $f : X \longrightarrow Y$ such that $fx \cdot fy = f(x \cdot y)$ for each couple x, y in X. If the groupoids are described as $\delta : FX = X \times X \longrightarrow X$, $\sigma : FY = Y \times Y \longrightarrow Y$, this equation can be written as $\sigma(f \times f) = f\delta$, in other words, as

$$(*) \qquad \sigma\, Ff = f\delta.$$

Analogously, bigroupoids (algebras with two binary operations) can be expressed as maps $FX \longrightarrow X$ and their homomorphisms are characterized by $(*)$ using the functor

$$FX = X \times X + X \times X, \quad Ff = f \times f + f \times f$$

where + denotes the disjoint union – the coproduct in the category of sets.

Actually, all algebras considered in the preceding paragraphs are special instances of functor algebras: Given a category **K** and a functor $F : \mathbf{K} \longrightarrow \mathbf{K}$, an *$F$-algebra* is a **K**-map $\delta : FX \longrightarrow X$ where the **K**-object X is called the underlying object. A homomorphism from an F-algebra $\delta : FX \longrightarrow X$ to an F-algebra $\sigma : FY \longrightarrow Y$ is a **K**-map $f : X \longrightarrow Y$ satisfying $(*)$. F-algebras and their homomorphisms from a category F-alg with a natural underlying functor to **K**.

Note that, in the examples of algebras considered above, **K** was the category Set of sets in case of classical or infinitary algebras; it was Set × Set for two sorted algebras (more generally: Set^S for S-sorted algebras); and it was the category of Z-complete posets and their Z-continuous maps for Z-continuous algebras.

Not every variety of classical algebras is of the form F-alg. However, varieties of functor algebras can be defined to include all classical cases. In general, reasonable properties of the category of functor algebras and their varieties are guaranteed if free algebras exist.

Given a functor $F : \mathbf{K} \longrightarrow \mathbf{K}$ a free F-algebra over a **K**-object X is an F-algebra $\phi : FX^\bullet \longrightarrow X^\bullet$ equipped with a **K**-map $\nu : X \longrightarrow X^\bullet$ with the following universal property: for every F-algebra $\delta : FY \longrightarrow Y$ and every **K**-map $f : X \longrightarrow Y$ there is a unique homomorphism $f^\bullet : X^\bullet \longrightarrow Y$ from $FX^\bullet \longrightarrow X^\bullet$ to $FY \longrightarrow Y$ with $f^\bullet\nu = f$.

Free F-algebras need not exist. A typical example is provided by the functor $F = \mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$ – the (covariant) power set functor

$$\mathcal{P}X = \{Y \mid Y \subseteq X\}, \quad \mathcal{P}f(Y) = f(Y)$$

which does not admit a free algebra over any set X.

VI. Existence and construction of free algebras

There are essentially two methods to prove the existence and to construct free algebras. The former is based on the *bounded generation* principle. A typical theorem, in terms of functor algebras, is the following:

Let the category K be complete, wellpowered and co-wellpowered. If every K-object X generates, up to isomorphism, only a *set* of non-isomorphic F-algebras then the free F-algebra over X exists for every K-object X.

The latter method simulates the classical construction of absolutely free algebras using terms. The formulation for functor algebras is a follows.

Free algebra construction. Suppose $F : \mathbf{K} \longrightarrow \mathbf{K}$ is a functor where K is a complete category. Let X be a K-object. Define a chain $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots \longrightarrow X_i \longrightarrow \ldots$ (where i runs over ordinals) consisting of K-objects X_i and K-maps $\nu_{ij} : X_i \longrightarrow X_j$, using induction, as follows:

X_0	$=$	X;	
X_1	$=$	$X + FX_0$	(where $+$ denotes the coproduct) $\nu_{01} : X_0 \to X_1$ is the coproduct injection
X_2	$=$	$X + FX_1$;	$\nu_{12} = 1_X + F\nu_{01} : X_1 \to X_2$;
.			
.			
.			
X_ω	$=$	$X + \mathrm{colim}_{i<\omega} FX_i$	the colimit of the chain $FX_0 \to FX_1 \to \ldots \to FX_i \to \ldots$ $\nu_{i\omega} = 1_X + \gamma_{i\omega}$ where $\gamma_{i\omega} : FX_i \to \mathrm{colim}_{i<\omega} FX_i$ are the colimit maps.
X_i	$=$	$X + \mathrm{colim}_{j<i} FX_j$	for every ordinal i; the maps ν_{ij} are defined similarly as for $i = \omega$.

We say that the free algebra construction *stops* (or *converges*) if $\nu_{i,i+1} : X_i \longrightarrow X_{i+1}$ is an isomorphism for some i. One can prove that then $FX_i \longrightarrow X_{i+1} \longrightarrow X_i$ (where the former arrow is the coproduct injection and the latter is $(\nu_{i,i+1})^{-1}$), together with $X_0 \longrightarrow X_i$, is the free F-algebra over X.

A typical convergence theorem, this construction is based on, is the following:

If $F : \mathbf{K} \to \mathbf{K}$ preserves colimits of chains of length α (α a limit ordinal) then, for every X in K, the free algebra construction over X stops and yields the free F-algebra over X.

For details, finer variants and references, see [6], [18].

VII. The most general situation and the dualization

Let us consider the most general situation of *structures over a category* K where structures are not specified, i.e., we have only a category S the objects of which are thought as K-objects equipped with some structure and morphisms are K-maps preserving these structures. This is expressed by a faithful functor $U : \mathbf{S} \longrightarrow \mathbf{K}$. (An alternative approach to mathematical structures was investigated by the French school see e.g., Bourbaki [8], Ehresmann [11].) Then an object X^* of S is a *free* S-*structure over a* K-object X iff there is a K-morphism $\nu : X \longrightarrow UX^*$ (insertion of generators) such that for every S-structure S and every K-morphism $f : X \longrightarrow US$ there is a unique S-morphism $f^* : X^* \longrightarrow S$ with $Uf^* . \nu = f$. As we have seen, free structures need not exist, they do not exist even in reasonable cases. However, if a free S-structure X^* over X does exist for every K-object X, we obtain a pair of adjoint functors

$$\mathbf{S} \xrightarrow{U} \mathbf{K} \qquad \mathbf{K} \xrightarrow{*} \mathbf{S}$$

The investigation of adjoint functors is one of the main streams in category theory which is included in every textbook or monograph [24], [16], [2].

The famous Adjoint Functor Theorem is the abstract version of the construction of free structures based on bounded generation, mentioned above for special cases. The highly developed *theory of monads* (see, e.g., [24],[22]) handles algebras in this way – the existence of free algebras is already incorporated into the definition of monads and adjoint situations are the main tool of the theory. What is still reasonable to do in concrete cases (and we tried to persuade you into it) is to decide whether free structures do exist or not, and to find some construction of free structures (like the free algebra construction from the preceding paragraph) which not only gives their existence but also enables us to investigate their internal structure. The latter is important – let us recall the case of free test algebras: the fact that they are not separable cannot be derived from their existence – one needs their explicit description. Also, the existence of a free S-structure over a K-object X can depend on X; (e.g., for every cardinal α there exists a functor F_α : Set $\longrightarrow$ Set such that a free F_α-algebra over a set X exists iff card $X < \alpha$), in which case the 'global approach' using adjoint functors fails.

As a final remark let us mention *cofree* structures. The abstract categorical level has the advantage that everything can be dualized. The formal dualization of the notion of a free structure is that of a *cofree* structure. Though the cofree structures also appear naturally in mathematics and in computer science (e.g.. in connection with the observability of automata, see [6]), their role seems to be much more limited than that of free structures. Since adjoint situations are selfdual the abstract theory can be applied both to the free structures and to the cofree ones. However, in particular cases, there is no relation between free structures and cofree structures. Nevertheless, the unified approach makes our insight deeper and this is the main goal of category theory.

References

[1] ADJ (J.A. Goguen, J.W. Thatcher, E.E. Wagner, J.B. Wright): Initial algebra semantics and continuous theories, JACM 24 (1977), 68-95.

[2] J. Adámek, H. Herrlich, G.E. Strecker: Abstract and concrete categories, Wiley, New York 1990.

[3] J. Adámek, E. Nelson, J. Reiterman: Tree constructions of free continuous algebras, J. Comp. Syst. Sci. 24, (1982), 114-146.

[4] J. Adámek, E. Nelson: Separately continuous algebras. Theor. Comp. Sci. 27 (1983), 293-313.

[5] J. Adámek, E. Nelson, J. Reiterman: Continuous semilattices, Theor. Comp. Sci. 43(1986), 293-313.

[6] J. Adámek, V. Trnková: Automata und Algebras in Categories, Kluwer Academic Publishers, Dodrecht 1990.

[7] G. Birkhoff: On the structure of abstract algebras, Proc. Cambridge Phil. Soc. 29 (1935), 441-464.

[8] N. Bourbaki: Elements de Mathématique – Théorie des Ensembles, Hermann, Paris 1958.

[9] P.M. Cohn: Universal Algebra, Harper and Row, New York 1965.

[10] D.S. Scott: The lattice of flow diagrams, in: Semantics of algorithmic languages, Lecture Notes in Math. 188 (1971), 311-366.

[11] C. Ehresmann: Catégories et Structures, Dunod, Paris 1965.

[12] H. Ehrig, B. Mahr: Fundamentals of Algebraic Specification 1 – Equations and Initial Semantics, Springer-Verlag 1985.

[13] H. Gaifman: Infinite Boolean polynomials, Fund. Math. 54 (1964), 229-250.

[14] G. Grätzer: Universal Algebra, Van Nostrand, Princeton 1965.

[15] A.W. Hales: On the non-existence of free complete Boolean algebras, Fund. Math. 54 (1964), 45-66.

[16] H. Herrlich, G.E. Strecker: Category Theory, Allyn and Bacon, Boston 1973.

[17] D. Kozen: A representation theorem for models of $*$-free PDL, IBM Research, July 1979.

[18] M.G. Kelly: A unified treatment of transfinite constructions ..., Bull. Austral. Math. Soc. 22 (1982), 1-85.

[19] F.W. Lawvere: Functorial semantics of algebraic theories, Dissertation, Columbia University, New York 1963.

[20] F.E.J. Linton: Some aspects of equational theories, in: Proc. Conf. Categorical Algebra, La Jolla 1965, Springer-Verlag 1966, 84-94.

[21] E.G. Manes: A triple miscellany: some aspects of the theory of algebras over a triple, Dissertation, Wesleyan University 1967.

[22] E.G. Manes: Algebraic Theories, Springer-Verlag 1976.

[23] A.I. Malcev: Free topological algebras, Izv. Akad. Nauk. SSSR, Ser. Mat. 21(1957), 171-198 (Amer. Math. Soc. Transl. Ser. 2. 17 (1961), 173-200).

[24] S. MacLane: Categories for the Working Mathematician, Springer-Verlag 1971.

[25] H.E. Porst: On free topological algebras, Cahiers Top. Géom. Diff. Cat. 28(1987), 235-253.

[26] V.R. Pratt: Dynamic algebras; examples, constructions, applications, Laboratory Comp. Sci. Mass. Inst. of Technology. July 1979.

[27] J. Reiterman: The category of complete lattices as a category of algebras, Algebra Univ. 26(1989), 367-379.

[28] J. Reiterman, V. Trnková: Dynamic algebras with test, J. Comp. Syst. Sci. 35 (1987), 229-242.

[29] J. Slominski: The theory of abstract algebras with infinitary operations, Rozprawy Mat. 18 (1959).

[30] K. Segerberg: A completeness theorem in the modal logical of programs, Notices Amer. Math. Soc. 24 (1977), A-522.