Category Theory at Work
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261-273

Algebra $\subset$ Topology ?!

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ABSTRACT. In his thesis [2], H. Bargenda constructed universal (E,M)-algebraic hulls $V:\mathbf{A}\longrightarrow\mathbf{X}$ for right adjoints $U:\mathbf{C}\longrightarrow\mathbf{X}$, i.e., X carries an (E,M)-factorization structure for sources [6,1], M a conglomerate of mono-sources, V lifts (E,M)-factorizations uniquely [1, 20.23], is right adjoint, and is universal with these properties. The construction needs enough E-projectives in X, although in case of $UF(\mathbf{E})\subseteq\mathbf{E}$ (where $F\dashv U$) the E-reflective hull A generated by the image of the comparison functor in the associated Eilenberg-Moore category $\mathbf{X}^{UF}$ always serves as a universal hull. This paper deals with a common generalization of Bargenda's and the Eilenberg-Moore construction of A. Surprizingly, A lives in any M-topological [1, 21K.] extension $\mathbf{Y}\longrightarrow\mathbf{X}$ of U which may be illegitimate and, therefore, is always available [7, 8], subject to minor technical assumptions on U.

0. Introduction.

The papers "Algebra $\cap$ Topology = Compactness" [9] and "Algebra $\cup$ Topology" [10] are serious reasons for the question mark in the title. Nevertheless, we shall prove

$$\text{algebraic hulls} \subseteq \text{topological extensions},$$

therefore, the exclamation mark!

Throughout, let X denote a category together with a fixed (E, M)-factorization structure for sources in the sense of H. Herrlich [1, 6] and let M be a conglomerate of mono-sources. Furthermore, let $U:\mathbf{C}\longrightarrow\mathbf{X}$ denote a right adjoint functor (with left adjoint F) admitting a M-*topological* extension

$$\begin{array}{ccl} \mathbf{C} & \xhookrightarrow{\text{inclusion}} & \mathbf{Y}, \quad \text{full,} \\ {\scriptstyle F\overset{\eta}{\dashv}U}\downarrow & \swarrow\; |\;| & \text{M-topological} . \\ \mathbf{X} & & \end{array}$$

i.e., every $|\;|$-structured source in M has a unique $|\;|$-initial lift [1, 21K.].

Such a functor $U = |\;||_{\mathbf{C}}$ has to be faithful and *amnestic*, i.e., any C-isomorphism f whose underlying morphism $Uf = |f|$ is an identity, has to be an identity itself. These more technical assumptions are not serious, because for any right adjoint functor there exists a

This work is an original contribution and will not appear elsewhere.

universal faithful and amnestic modification, which is right adjoint, too (see also [1, 5.33]). In what follows, the M–topological extension above need not be legitimate [1, 2.3, 6.16]. However, it always exists in some (possibly higher) universe (take for instance E^2 in [7,8]).

H. Bargenda introduced $(\mathbf{E}, \mathbf{M})$-algebraic functors [2, 3] as a generalization of H. Herrlich's regular functors [5], in order to cover underlying space functors of universal topological algebras, which sometimes fail to be regular [12, 13]. A right adjoint functor $V : \mathbf{A} \longrightarrow \mathbf{X}$ is called $(\mathbf{E}, \mathbf{M})$-*algebraic*, if it lifts $(\mathbf{E}, \mathbf{M})$-factorizations of all sources $V((f_i : A \longrightarrow A_i)_I)$ uniquely. If this is the case, $\mathbf{A}$ turns out to be a $(V^{-1}(\mathbf{E}), V^{-1}(\mathbf{M}))$-category, the morphisms in $V^{-1}(\mathbf{E})$ or the sources in $V^{-1}(\mathbf{M})$ being V-final or V-initial, respectively.

In the presence of enough E–projectives in $\mathbf{X}$ there exists an universal $(\mathbf{E}, \mathbf{M})$-algebraic hull $V : \mathbf{A} \longrightarrow \mathbf{X}$ for each right adjoint $U : \mathbf{C} \longrightarrow \mathbf{X}$. The same holds, if $UF(\mathbf{E}) \subseteq \mathbf{E}$ (where $F \dashv U$) without any additional assumption on $\mathbf{X}$. In this case, the E–reflective hull generated by the image of the comparison functor in the associated Eilenberg–Moore category $\mathbf{X}^{UF}$ solves the problem [2,3,4].

Section 1 contains a common generalization of both constructions divided into two steps. First of all, one gets a canonical extension $U' : \mathbf{C}' \longrightarrow \mathbf{X}$ of U and a left adjoint F' for U' with $U'F'(\mathbf{E}) \subseteq \mathbf{E}$. After that, the Eilenberg–Moore machinery applies.

Alternatively, there is a purely topological construction of the $(\mathbf{E}, \mathbf{M})$-algebraic hull V in section 2 as the restriction of $|\ | : \mathbf{Y} \longrightarrow \mathbf{X}$ to a certain subcategory $\mathbf{A}$, justifying the title of the paper. This has the advantage of unifying both, $\mathbf{C}$ and $\mathbf{A}$, in a common extension $\mathbf{Y}$. Considering $F \dashv U$ and $G \dashv V$ as functors $F, G : \mathbf{X} \longrightarrow \mathbf{Y}$, one may ask for natural transformations between them. In fact, there is a canonical one from G to F.

For standard notations, terminology, and results we refer to [1, 11].

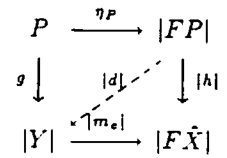
1. Coreflectivity of E–monads.

In general, UF fails to preserve the E–class. Therefore, consider for every $e : X \longrightarrow \hat{X} \in \mathbf{E}$

$$Fe = m_e s_e, \quad |s_e| \in \mathbf{E}, \ |m_e| \in \mathbf{M},$$

and $m_e : Y \longrightarrow F\hat{X}$ the initial lift of $|m_e|$. Choose $m_e = idF\hat{X}$ if $UF_e \in \mathbf{E}$.

1.1. Lemma. Let P be an E–projective $\mathbf{X}$–object. Then η_P is $|\ |$-*orthogonal* to each m_e, $e \in \mathbf{E}$, i.e., for every commutative square



there exists an unique diagonal $d : FP \longrightarrow Y$ in Y as indicated, rendering the triangles commutative.

Proof. By assumption on P, g factorizes via $|s_e|$, $g = |s_e|f$, and $f : P \longrightarrow UFX$ admits an extension $\overline{f} : FP \longrightarrow FX$, $U\overline{f}\eta_P = f$. Now take $d := s_e\overline{f}$. Then $|d|\eta_P = |s_e|f = g$ and

$$U(m_e d)\eta_P = U((Fe)\overline{f})\eta_P = |m_e||s_e|f = |m_e|g = Uh\eta_P,$$

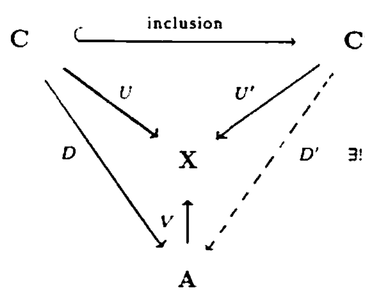
hence $m_e d = h$.

□

If $UF(\mathrm{E}) \subseteq \mathrm{E}$, then every m_e is an isomorphism, and for any X–object X, η_X is $|\ |$–orthogonal to each m_e, $e \in \mathrm{E}$. Therefore, in this section we require the following general

1.2. Assumption. There are *enough* objects P in X, such that η_P is $|\ |$–orthognal to each m_e, i.e., for every X–object X there exists such a $P = P_X$ together with $e_X : P_X \longrightarrow X \in \mathrm{E}$.

1.3. Theorem. There exists a legitimate extension $\mathbf{C}' \subseteq \mathbf{Y}$ of C such that $U' := |\ ||_{\mathbf{C}'}$ has a left adjoint F' and $U'F'(\mathrm{E}) \subseteq \mathrm{E}$. Moreover, any concrete functor $D : (\mathbf{C}, U) \longrightarrow (\mathbf{A}, V)$ to an (E, M)–algebraic category $(\mathbf{A}, V)$ has a unique concrete factorization via the inclusion $\mathbf{C} \hookrightarrow \mathbf{C}'$:



Proof. Let $\mathbf{Z} \subseteq \mathbf{Y}$ denote the full subcategory generated by all Y–objects Z with the following properties:

1. The source of all Y-morphisms $Z \longrightarrow C \in \mathbf{C}$ is $|\ |$-initial and its underlying source is in $\mathbf{M}$.
2. Each X-morphism $g : P \longrightarrow |Z|$, P as in 1.2., admits a unique extension $\overline{g} : FP \longrightarrow Z$, i.e.,

$$\begin{array}{ccc} P & \xrightarrow{\eta P} & UFP = |FP| \\ & {\scriptstyle g}\searrow \quad \swarrow {\scriptstyle |\overline{g}|\ \exists!} & \\ & |Z| & \end{array} .$$

Obviously, $\mathbf{C} \subseteq \mathbf{Z}$. Next we construct a left adjoint $G : \mathbf{X} \longrightarrow \mathbf{Z}$ for $|\ ||_{\mathbf{Z}}$ using the factorizations $Fe = m_e s_e$ above:

$$\begin{array}{ccc} FP_X & \xrightarrow{Fe_X} & FX\ , \quad e_X \text{ as in 1.2.,} \\ & {\scriptstyle s := s_{e_X}}\searrow \quad \nearrow {\scriptstyle m_{e_X} =: m} & \\ & GX & \end{array} .$$

Then GX fulfills property 1., because m is the initial lift of $|m| \in \mathbf{M}$. Furthermore, for each X-morphism $g : P \longrightarrow |GX|$, P as in 1.2., there is an unique C-morphism $h : FP \longrightarrow FX$ rendering the following rectangle commutative:

$$\begin{array}{ccc} P & \xrightarrow{\eta_P} & UFP \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle |\overline{g}|} & \downarrow {\scriptstyle Uh=|h|} \\ |GX| & \xrightarrow{|m|} & UFX = |FX| \end{array} .$$

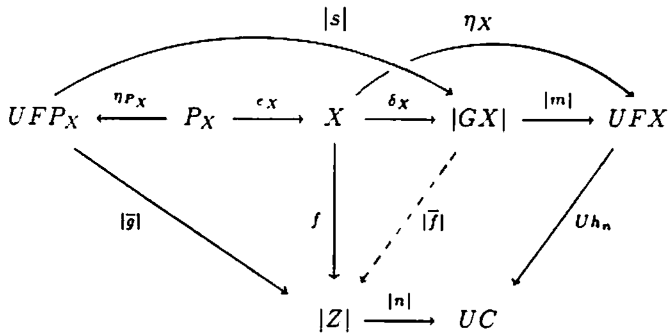
Therefore, the diagonal $\overline{g} : FP \longrightarrow GX$ exists. Moreover, it turns out to be uniquely determined by $|\overline{g}|\eta_P = g$, because h is unique and $|m| \in M$. This proves property 2. for GX, hence $GX \in \mathbf{Z}$.

Futhermore, there is a canonical arrow $\delta_X : X \longrightarrow |GX|$ as diagonal in

$$\begin{array}{ccc} P_X & \xrightarrow{e_X} & X \\ {\scriptstyle \eta_{P_X}}\downarrow & & \\ UFP_X & \swarrow {\scriptstyle \delta_X} & \downarrow {\scriptstyle \eta_X} \\ {\scriptstyle |s|}\downarrow & & \\ |GX| & \xrightarrow{|m|} & UFX \end{array} .$$

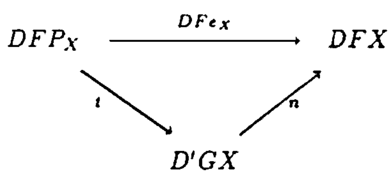
Now consider any X–morphism $f : X \longrightarrow |Z|$ with $Z \in \mathbf{Z}$ and $g := fe_X$. By condition 2., there exists an unique $\bar{g} : FP_X \longrightarrow Z$ with $|\bar{g}|\eta_P = g$. Furthermore, for any morphism $n : Z \longrightarrow C \in \mathbf{C}$ there is an unique extension $h_n : FX \longrightarrow C$ of $|n|f$ with $Uh_n\eta_X = |n|f$.

The following diagram commutes



Hence the diagonal $\bar{f} : GX \longrightarrow Z$ exists, because $|s| \in \mathbf{E}$, $(|n|) \in \mathbf{M}$, and (n) initial. Its uniqueness is obvious. This shows $G \overset{\delta}{\dashv} |\ ||_{\mathbf{Z}}$.

Since $\mathbf{C} \subseteq \mathbf{Z}$ we get a canonical isomorphism $GX \cong FX$ if GX is isomorphic to some C–object. Therefore, choose $GX = FX$ and $\delta_X = \eta_X$ in this case, $\mathbf{C}' := \mathbf{C} \cup G(\mathbf{X}) \subseteq \mathbf{Y}$, full, $U' := |\ ||_{\mathbf{C}'}$, and $F' := G$. Now let $D : (\mathbf{C}, U) \longrightarrow (\mathbf{A}, V)$ where $(\mathbf{A}, V)$ is $(\mathbf{E}, \mathbf{M})$–algebraic. For GX, $X \in \mathbf{X}$, consider the unique lift of $VDFe_X = UFe_X = |m||s|$ along V:



$Vt = |s|, Vn = |m|$.

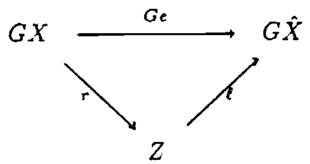
Then $D'GX = DGX$ for $GX \in \mathbf{C}$, because $m = id_{FX}$ in this case. Together with $D'C := C$ for other $C \in \mathbf{C}$, this defines $D' : \mathbf{C}' \longrightarrow \mathbf{A}$ on objects.

For $\mathbf{C}'$–morphisms $f : C \longrightarrow GX$, $C \in \mathbf{C}$, consider $g := mf$, which is a C–morphism. Now Dg is an A–morphism, such that $VDg = Ug$ factorizes via $|m| = Vn$, n being V–initial. This yields an unique A–morphism $D'f : D'C \longrightarrow D'GX$ with $VD'f = U'f$.

For $\mathbf{C}'$–morphisms $f : GX \longrightarrow C$, $C \in \mathbf{C}$, consider $g := fs$ and use the finality of t.Combination of both yields $D'f : D'GX' \longrightarrow D'GX$ with $VD'f = U'f$ for the remaining case $f : GX' \longrightarrow GX$ in $\mathbf{C}'$.

As an (E, M)-algebraic functor, V is faithful and reflects identities. Therefore, the equation $VD' = U'$ forces D' to be functorial. For similar reasons, D' is unique.

It remains to show, that $(U'F')(e) \in \mathbf{E}$ if $e : X \longrightarrow \hat{X} \in \mathbf{E}$. To prove this, consider



in $\mathbf{Y}$, $|r| \in \mathbf{E}$, $|\ell| \in \mathbf{M}$,

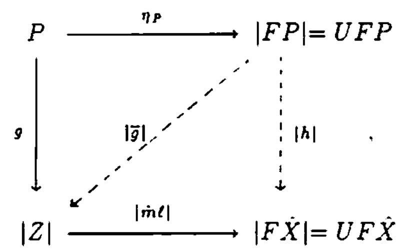
and ℓ initial with respect to $|\ |$. Then Z fulfills condition 1. above. For 2. abbreviate $\hat{m} := m_{e_{\hat{X}}}$ and observe

$$|Fe||m|\delta_X = |Fe|\eta_X = \eta_{\hat{X}}e = |\hat{m}|\delta_{\hat{X}}e = |\hat{m}||Ge|\delta_X,$$

hence $(Fe)m = \hat{m}(Ge)$ and so

$$F(ee_X) = FeFe_X = (Fe)ms = \hat{m}(Ge)s = (\hat{m}\ell)(rs),$$

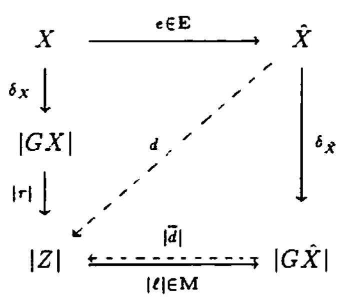
thus $\hat{m}\ell$ equals m_{ee_X} up to an isomorphism. The square



, $Uh\eta_P = |\hat{m}\ell|g$,

commutes for any P as in 1.2. and each $\mathbf{X}$–morphism g. Therefore, the diagonal $\overline{g} : FP \longrightarrow Z$ always exists as indicated. Its uniqueness follows from $|\overline{g}|\eta_P = g$ using uniqueness of h and the monomorphism $|\hat{m}\ell|$. This proves condition 2. for Z.

The following rectangle commutes in $\mathbf{X}$:



Hence there exists the diagonal d as indicated and its extension $\bar{d}$. Futhermore,

$$|\ell||\bar{d}|\delta_{\dot{X}} = |\ell|d = \delta_{\dot{X}}$$

implies $\ell\bar{d} = id_{G\dot{X}}$, thus ℓ turns out to be an isomorphism and $|Ge| = (U'F')(e) \in \mathrm{E}$.

□

1.4. Remark. The construction of G above yields also a natural transformation $\nu : F' \longrightarrow F$, which is pointwise initial and in $|\ |^{-1}(\mathrm{M})$. Choosing $e_P = id_P$ for η_P orthogonal to each m_e, ν_P turns out to be an isomorphism and vice versa.

In terms of monads and their morphisms 1.3. translates as follows:

1.5. Corollary. Let $(T = UF, \eta, \mu = U\varepsilon_F)$ be the monad defined by the adjunction $F \underset{\varepsilon}{\overset{\eta}{\dashv}} U$. Then there exists an E–monad (T', η', μ') on $\mathbf{X}$ together with a couniversal monad–morphism $\nu : T' \longrightarrow T$, which is pointwise in M.

Moreover, 1.3. yields a quick proof for existence of the (E, M)–algebraic hull, applying the Eilenberg–Moore construction to $T' = U'F'$ [4].

1.6. Corollary. $U : \mathbf{C} \longrightarrow \mathbf{X}$ posesses an universal (E, M)–algebraic hull $E : (\mathbf{C}, U) \longrightarrow (\mathbf{A}, V)$.

Proof. According to 1.3. the universal (E, M)–algebraic hulls of U and U' coincide. Using wellknown properties of the Eilenberg–Moore construction [4], the latter turns out to be the E–reflective hull generated by the image of the comparison functor $\mathbf{C}' \longrightarrow \mathbf{X}^{U'F'}$.

□

2. A topological construction of algebraic hulls.

The main result of section 1 enables one to assume that $UF(\mathrm{E}) \subseteq \mathrm{E}$. Just as in the proof of 1.3. consider $\mathbf{Z} \subseteq \mathbf{Y}$ with respect to all $P \in \mathbf{C}$.

2.1. Lemma. $\mathbf{Z}$ is closed under the formation of $(|\ |^{-1}(\mathrm{E})$, initial $|\ |^{-1}(\mathrm{M}))$–factorizations.

Proof. By construction, $F \underset{\varepsilon}{\overset{\eta}{\dashv}} |\ ||_{\mathbf{Z}}$. Consider

$$(Z \xrightarrow{r} Y \xrightarrow{m_i} Z_i)_I,$$

$Z, Z_i \in \mathbf{Z}, i \in I, |r| \in \mathbf{E}, (|m_i|)_I \in \mathbf{M}, (m_i)$ initial. Then Y fulfills condition 1. for Z–objects. To prove 2., observe that the following squares commute:

$$\begin{array}{ccccc}
UF|Z| = |F|Z|| & \xrightarrow{UF|r| \in \mathbf{E}} & & |F|Y|| = UF|Y| & \\
|\varepsilon_Z| \downarrow & & \swarrow & \downarrow |F|m_i|| & \\
|Z| & |d| & & |F|Z_i|| & , \quad i \in I \; . \\
|r| \downarrow & \swarrow & & \downarrow |\varepsilon_{Z_i}| & \\
|Y| & \xrightarrow[|m_i|]{} & & |Z_i| &
\end{array}$$

The existence of $d : F|Y| \longrightarrow |Y|$ as indicated follows from $UF|r| \in \mathbf{E}$, $(|m_i|)_I \in \mathbf{M}$, and $(m_i)_I$ initial.

Now take any X–morphism $f : X \longrightarrow |Y|$ and choose $\overline{f} := dFf : FX \longrightarrow Y$. Then

$$\begin{aligned}
|m_i||\overline{f}|\eta_X &= |m_i||d||Ff|\eta_X = |\varepsilon_{Z_i}||F|m_i||Ff|\eta_X = |\varepsilon_{Z_i}|\eta_{|Z_i|}|m_i|f \\
&= |m_i|f,
\end{aligned}$$

for all $i \in I$, hence $|\overline{f}|\eta_X = f$. Furthermore, $\overline{f}$ is uniquely determined by the latter equation, according to $(|m_i|)_I \in \mathbf{M}$ and universality of η_X.

□

2.2. Lemma. Let $r : Z \longrightarrow \overline{Z}$ be a Z–morphism with $|r| \in \mathbf{E}$. Then there exists a $|\;|_{\mathbf{Z}}$–final lift $\hat{r} : Z \longrightarrow \hat{Z}$, $|\hat{r}| = |r|$.

Proof. For each C–object C consider the set

$$\begin{aligned}
H_C = \{h | h &\text{ is an X–morphism and } h|r| = |g_h| \text{ for some Z–morphism } g_h : Z \to C\} \\
&\subseteq hom(|\overline{Z}|, |C|).
\end{aligned}$$

The source

$$(|\overline{Z}|, h)_{h \in H_C, C \in \mathbf{C}}$$

contains every $|m|$, $m : \overline{Z} \longrightarrow C \in \mathbf{C}$ a Z–morphism. According to condition 1. on Z the source above belongs to M [6]. For its $|\;|$–initial lift in Y,

$$(\hat{Z}, \hat{h} : \hat{Z} \longrightarrow C),$$

there exists a Y–morphism $\hat{r} : Z \longrightarrow \hat{Z}$ such that $|\hat{r}| = |r|$ and $\hat{h}\hat{r} = g_h$ for all h, hence by 2.1., $\hat{Z} \in \mathbf{Z}$.

Now consider any X–morphism $f : |\hat{Z}| \longrightarrow |\tilde{Z}|, \tilde{Z} \in \mathbf{Z}$, such that

$$f|\hat{r}| = |g_f|,\ g_f : Z \longrightarrow \tilde{Z} \quad \text{a Z–morphism},$$

and the initial monosource of all Z–morphisms

$$n : \tilde{Z} \longrightarrow C \in \mathbf{C}.$$

Then $h_n := |n|f$ has a lift $\hat{h}_n : \hat{Z} \longrightarrow C$ with

$$\hat{h}_n\hat{r} = ng_f,$$

hence

$$|\hat{h}_n||\hat{r}| = |n||g_f| = |n|f|\hat{r}| \Rightarrow |\hat{h}_n| = |n|f.$$

Initiality of the source (n) yields a Z–morphism

$$\hat{f} : \hat{Z} \longrightarrow \tilde{Z} \quad \text{with} \quad \hat{h}_n = n\hat{f},$$

hence

$$ng_f = \hat{h}_n\hat{r} = n\hat{f}\hat{r} \quad \text{for all } n,$$

consequently

$$g_f = \hat{f}\hat{r} \quad \text{and} \quad |\hat{f}| = f.$$

□

2.3. Corollary. Let $s : T \longrightarrow X$ be a morphism in X such that $|Fs| \in \mathbf{E}$. Then Fs is final with respect to $|\ ||_{\mathbf{Z}}$.

Proof. Consider the $|\ ||_{\mathbf{Z}}$– final lift $\hat{e} : FT \longrightarrow \hat{Z}$ of $|Fs|$. Then $id_{|FX|}$ has a lift $b : \hat{Z} \longrightarrow FX$, $b\hat{e} = Fs$, and there exists a Z–morphism $c : FX \longrightarrow \hat{Z}$, such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & |FX| \\ & \searrow^{\eta_X} & \downarrow |c| \\ & & |FX| = |\hat{Z}| \end{array} \qquad \text{commutes}$$

Furthermore, $|b||c|\eta_X = \eta_X$, hence $bc = id_{FX}$, thus b is an isomorphism.

□

Next we consider for every **Z**-object Z its counit

$$\varepsilon_Z : F|Z| \longrightarrow Z$$

and the $|\ ||_{\mathbf{Z}}$-final lift of $|\varepsilon_Z|$, thus getting

$$\hat{\varepsilon}_Z : F|Z| \longrightarrow \hat{Z} \quad \text{final}, \quad |\hat{\varepsilon}_Z| = |\varepsilon_Z|.$$

The **Z**-objects $\hat{Z}$ constitute a full legitimate subcategory $\mathbf{A} \subseteq \mathbf{Z}$. Moreover, the identities $id_{|Z|} = id_{|\hat{Z}|} : |\hat{Z}| \longrightarrow |Z|$ have a unique lift

$$b_Z : \hat{Z} \longrightarrow Z, \quad b_Z\hat{\varepsilon}_Z = \varepsilon_Z.$$

Abbreviate the restriction of $|\ |$ to **A** by $V : \mathbf{A} \longrightarrow \mathbf{X}$.

2.4. Proposition. The following holds for **A** :

(a) $F(\mathbf{X}) \subseteq \mathbf{A}$, hence $F \underset{\varepsilon}{\overset{\eta}{\dashv}} V(= |\ ||_{\mathbf{A}})$ with the same unit η and counit ε as in case of $F \underset{\varepsilon}{\overset{\eta}{\dashv}} |\ ||_{\mathbf{Z}}$, restricted to **A**.

(b) **A** is bicoreflective in **Z** with coreflections $b_Z : \hat{Z} \longrightarrow Z$.

(c) For each $A \in \mathbf{A}$, the source of all **A**-morphisms $A \longrightarrow \hat{C}, C \in \mathbf{C}$, belongs to $V^{-1}(\mathbf{M})$.

(d) All morphism in $V^{-1}(\mathbf{E})$ are $|\ ||_{\mathbf{Z}}$-final.

Proof. For each $X \in \mathbf{X}$ the following holds

$$|\varepsilon_{FX}||F\eta_X|\eta_X = |\varepsilon_{FX}|\eta_{|FX|}\eta_X = \eta_X \quad \Rightarrow \quad \varepsilon_{FX}F\eta_X = id_{FX},$$

hence ε_{FX} is a retraction, thus final and $\widehat{FX} = FX$ as well as $b_{FX} = id_{FX}$. This proves (a).

By construction, $\varepsilon_A : F|A| \longrightarrow A$, $A \in \mathbf{A}$, is final, because for $Z \in \mathbf{Z}$ with $\hat{Z} = A$

$$\begin{aligned} &|\hat{\varepsilon}_Z| = |\varepsilon_Z| \text{ and } |\hat{Z}| = |Z| \\ \Rightarrow &|\hat{\varepsilon}_Z|\eta_{|F|\hat{Z}||} = |\varepsilon_Z|\eta_{|F|Z||} = id_{|F|Z||} = id_{|F|\hat{Z}||} = |\varepsilon_{\hat{Z}}|\eta_{|F|\hat{Z}||} \\ \Rightarrow &\hat{\varepsilon}_Z = \varepsilon_{\hat{Z}} : F|\hat{Z}| = F|Z| \longrightarrow \hat{Z}. \end{aligned}$$

Now consider some Z–morphism $f : A \longrightarrow Z$, $A \in \mathbf{A}$. Then the following rectangle commutes, according to $\varepsilon_Z = b_Z \hat{\varepsilon}_Z$

$$\begin{array}{ccccc} F|Z| & \xrightarrow{\hat{\varepsilon}_Z} & \hat{Z} & \xrightarrow{b_Z} & Z \\ {\scriptstyle F|f|}\uparrow & & & \nwarrow & \uparrow{\scriptstyle f} \\ F|A| & & \xrightarrow[\varepsilon_A]{} & & A \end{array} .$$

The dotted arrow exists by means of $|b_Z| = id_{|Z|}$ and finality of ε_A. This proves (b).

(c) is immediate from (b) and condition 1. for Z.

Let $s : A \longrightarrow B$ be in $V^{-1}(\mathbf{E})$. Then $|F|s|| \in \mathbf{E}$, hence $F|s|$ is final, by 2.3., and the following diagram commutes

$$\begin{array}{ccccl} F|A| & \xrightarrow{F|s|} & F|B| & , & \text{final}, \\ {\scriptstyle \varepsilon_A}\downarrow & & \downarrow{\scriptstyle \varepsilon_B} & , & \text{final} \\ A & \xrightarrow{s} & B & & \end{array} .$$

Hence s is final.

□

By construction, the restriction E of the coreflector $H : \mathbf{Z} \longrightarrow \mathbf{A}$ to $\mathbf{C}$ fulfills $VE = U$.

2.5. Theorem. $(\mathbf{A}, V)$ together with E is a universal $(\mathbf{E}, \mathbf{M})$-algebraic hull of $(\mathbf{C}, U)$.

Proof. By 2.4.(a), V is right adjoint. Consider any source $(A \xrightarrow{f_i} A_i)_I$ in $\mathbf{A}$. An $(\mathbf{E}, \mathbf{M})$-factorization of the corresponding underlying source in $\mathbf{X}$ admits a lift to $\mathbf{Z}$, using 2.1.. Application of the coreflector H yields a $(V^{-1}(\mathbf{E}), V^{-1}(\mathbf{M}))$ factorization of (f_i) in $\mathbf{A}$, which is uniquely determined by its underlying $(\mathbf{E}, \mathbf{M})$-factorization, according to 2.4.(d).

Now let $(\mathbf{B}, W)$ be $(\mathbf{E}, \mathbf{M})$-algebraic over $\mathbf{X}$ and $K : \mathbf{C} \longrightarrow \mathbf{B}$ a functor with $WK = U$. For each Z–object Z there is a canonical $(|\ |^{-1}(\mathbf{E})$, initial $|\ |^{-1}(\mathbf{M}))$-factorization

$$F|Z| \xrightarrow{\varepsilon_Z} Z \xrightarrow{m} C \in \mathbf{C}$$

of the C–source $(F|Z| \xrightarrow{m\varepsilon_Z} C)$, m running through all Z–morphisms $Z \longrightarrow C \in \mathbf{C}$. The unique lift of the $(\mathbf{E}, \mathbf{M})$-factorization

$$|F|Z|| \xrightarrow{|\varepsilon_Z|} |Z| \xrightarrow{|m|} |C|$$

of $(|m\varepsilon_Z|) = (Um\varepsilon_Z) = (W(Km\varepsilon_Z))$ along W defines an assignment on objects

$$Z \mapsto LZ \in \mathbf{B}, \quad WLZ = |Z|,$$

which extends K, because id_Z is one of the m's, if $Z \in \mathbf{C}$. Using the $(W^{-1}(\mathbf{E}), W^{-1}(\mathbf{M}))$-diagonalization property in $\mathbf{B}$, there is an assignment on morphisms

$$Z' \xrightarrow{f} Z \mapsto LZ' \xrightarrow{Lf} LZ$$

with $WLf = |f|$, extending K as well. $(\mathbf{E}, \mathbf{M})$-algebraicity of W and $WLf = |f|$ force L to be functorial and unique.

It remains to show, that L factorizes via H. This is automatically unique, because H is surjective on morphisms, although not full, in general. Therefore, consider an $\mathbf{A}$-morhism $h : \hat{Z}' \longrightarrow \hat{Z}$, and a $\mathbf{Z}$-morphism $f : Z' \longrightarrow Z$ with $Hf = h$, i.e.,

$$\begin{array}{ccc} \hat{Z}' & \xrightarrow{b_{Z'}} & Z' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \hat{Z} & \xrightarrow{b_Z} & Z \end{array} \quad \text{commutes} \quad .$$

where $|b_{Z'}| = WLb_{Z'}$ and $|b_Z| = WLb_Z$ are identities. But W is uniquely transportable, hence, $Lb_{Z'}$ and Lb_Z are identites and $Lf = Lh$. This shows $Lf = Lg$ for any two $\mathbf{Z}$-morphisms f, g with $Hf = Hg =: h$, thus L factorizes via H. For the same reasons as in case of L, this factorization is functorial.

□

Remark. In general, the associated Eilenberg-Moore category cannot be recovered as a full extension of $\mathbf{A}$ in $\mathbf{Y}$, even for the underlying set functor $|\ | : \mathbf{Top} \longrightarrow \mathbf{Set}$ of the category of topological spaces [14].

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