Category Theory at Work
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Axiomatizing the Category of Compact Hausdorff Spaces

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ABSTRACT. The process mentioned in the title includes a description of the varietal theory of compact Hausdorff spaces in the sense of F.E.J. Linton [12]. This follows from an internal axiomatization of the Stone–Čech–compactification of discrete spaces, regarded as endofunctor β: **Set** $\longrightarrow$ **Set** of the category of sets. Since the ultrafilter functor U: **Set** $\longrightarrow$ **Set** fulfills the same axioms as β, both coincide up to a natural isomorphism even as monads. This reproves a wellknown result due to E.G. Manes [13,14]. Recall R. Börger's nice external characterization of U as being terminal among all endofunctors T of **Set** preserving finite coproducts [4]. Now there are criteria available for injectivity or surjectivity of the unique natural transformation $T \longrightarrow U$, corresponding to Hausdorffness or compactness of a certain topology on T, respectively.

0. Introduction

The Stone–Čech–compactification βX of any set (discrete space) X is wellknown to be a Stone space (compact, T_0 and zero–dimensional) [11]. Therefore it may be regarded in a canonical way as closed subspace of the 2^X–th power of the two element discrete space 2,

$$\beta X = \overline{\beta X} \subseteq 2^{2^X},$$

with respect to the product–topology, i.e., the initial topology given by the projections $p_f : 2^{2^X} \longrightarrow 2, f \in 2^X$. The universal injection

$$\beta_X : X \longrightarrow \beta X$$

is given by $p_f \beta_X := f, f \in 2^X$, and we have

$$\beta X = \overline{\beta_X(X)}.$$

In section 1, we consider endofunctors T of the category **Set** of sets, which preserve finite coproducts (including the empty one). They admit a canonical factorization via the underlying set functor

$$\mathbf{Top} \longrightarrow \mathbf{Set}$$

This work is an original contribution and will not appear elsewhere.

of the category **Top** of topological spaces. Using this, we show independently from R. Börger [4], that β is terminal among all such T.

Section 2 investigates the pointwise injectivity of the unique natural transformation $T \longrightarrow \beta$. It turns out to correspond to Hausdorffness of the canonical topologies introduced in section 1. Section 3 deals with the pointwise surjectivity of $T \longrightarrow \beta$, which corresponds to compactness.

Combining both, bijectivity is characterized by internal properties of T in section 4. Since the varietal theory of compact Hausdorff spaces in the sense of F.E.J. Linton [12] is just

$$\beta^{op} : \mathbf{Set}^{op} \longrightarrow \beta(\mathbf{Set})^{op} \subseteq \mathbf{Top}^{op}, \quad \text{full},$$

we get an internal description of this from its dual β, simplified by means of the algebraic properties of varietal theories. This enables to axiomatize **HComp**, the category of compact Hausdorff spaces. Moreover, the ultrafilter functor

$$U : \mathbf{Set} \longrightarrow \mathbf{Set}$$

shares the characteristic properties of β, hence $U \cong \beta$ [13, 14].

For standard notations, terminology, and results see [1,10].

1. Terminality of β

Throughout this section let T and T' denote endofunctors of **Set** preserving finite coproducts. R. Börger [4] proved, that there is a unique natural transformation $\kappa : T \longrightarrow U$, namely

$$\kappa_X(x) := \{S \subseteq X | x \in T(S \hookrightarrow X)(TS)\}$$

for all sets X and $x \in TX$. Unfortunately, it is by no means obvious, which internal properties of T correspond to pointwise injectivity of κ or surjectivity, respectively. Therefore, we change from the ultrafilter side to the topological point of view.

1.1. Lemma. $\{T(S \hookrightarrow X)(TS) \mid S \subseteq X\}$ is a basis of clopen sets in TX. Moreover, any $Tf : TX \longrightarrow TY$, $f : X \longrightarrow Y$ a map, becomes continuous with respect to the corresponding topologies. The same holds for the components $\nu_X : T'X \longrightarrow TX$ of any natural transformation $\nu : T' \longrightarrow T$. In case of $T = \beta$, the topology mentioned above is just the usual one.

1.2. Lemma. $D := \bigcup_{x \in X} T(\{x\} \hookrightarrow X)(T\{x\})$ is dense in TX.

1.3. Corollary. Assume that every TX is Hausdorff. Then there exists at most one natural transformation $\nu : T' \longrightarrow T$.

Proof. By construction, $T1$, $1 := \{0\}$, is indiscrete, hence $T1 = \emptyset$ or $T1 \cong 1$ by assumption. This implies $\nu_{\{x\}} = \mu_{\{x\}}$ for all natural transformations $\nu, \mu : T' \longrightarrow T$ and any singleton $\{x\} \subseteq X$. Therefore ν_X and μ_X coincide on the dense subset $D' \subseteq T'X$ (1.2.). Moreover, they are continuous (1.1.), thus equal by assumption on TX.

□

1.4. Theorem. There exists exactly one natural transformation $\lambda : T \longrightarrow \beta$.

Proof. According to 1.1. and 1.3., it suffices to prove existence. To do this, we construct a natural transformation σ from T to the double powerset functor $2^{2^{-}}$ which factorizes via β. Using $2 = \{0\} \coprod \{1\}$ and the canonical maps $c : T\{0\} \longrightarrow \{0\}$, $d : T\{1\} \longrightarrow \{1\}$ as well as the canonical isomorphism $T2 \cong T\{0\} \coprod T\{1\}$ we get σ_X as the unique map rendering the upper rectangles in the following diagrams commutative

$$
\begin{tikzcd}
T2 \cong T\{0\} \coprod T\{1\} \arrow[r, "c \coprod d"] \arrow[r, bend left=60, "e"] & 2 & \\
TX \arrow[u, "Tf"] \arrow[r, dashed, "\sigma_X"] & 2^{2^X} \arrow[u, "p_f"'] & , \quad f \in 2^X, \\
TY \arrow[u, "Th"] \arrow[r, "\sigma_Y"'] & 2^{2^Y} \arrow[u, "2^{2^h}"'] & , \quad h : Y \longrightarrow X.
\end{tikzcd}
$$

The lower rectangles do commute for any map $h : Y \longrightarrow X$. Hence, $\sigma : T \longrightarrow 2^{2^{-}}$ is natural and, moreover, pointwise continuous by construction. Therefore, it suffices to show $\sigma_X(D) \subseteq \beta X$, where $D \subseteq TX$ is defined as in 1.2. To do this we abbreviate $T_x := T(\{x\} \hookrightarrow X)(T\{x\})$, $x \in X$, and prove $p_f \sigma_X(T_x) \subseteq \{f(x)\}$, $f \in 2^X$, thus $\sigma_X(T_x) \subseteq \beta_X(X) \subseteq \beta X$:

For $x \in f^{-1}(0) \subseteq X$ we have

$$T_x \subseteq T(f^{-1}(0) \hookrightarrow X)(T(f^{-1}(0))),$$

hence

$$Tf(T_x) \subseteq T(\{0\} \hookrightarrow 2)(T\{0\}),$$

because

$$\begin{array}{ccc} Tf^{-1}(0) & \xrightarrow{T(f|_{f^{-1}(0)})} & T\{0\} \\ \downarrow{\scriptstyle T(f^{-1}(0)\hookrightarrow X)} & & \downarrow{\scriptstyle T(\{0\}\hookrightarrow 2)} \\ TX & \xrightarrow{Tf} & T2 \end{array}$$

commutes. Moreover,

$$\begin{aligned} \underline{p_f\sigma_X(T_x)} &= eTf(T_x) \\ &\subseteq eT(\{0\} \hookrightarrow 2)(T\{0\}) \\ &= c(T\{0\}) = \{0\} = \underline{\{f(x)\}} \end{aligned}$$

since $eT(\{0\} \hookrightarrow 2) = c$ by construction. The same holds for $x \in f^{-1}(1) \subseteq X$.

□

1.5. Corollary. (cf. R. Börger [4], 2.3.)

(i) There exist unique natural transformations $\eta : id_{\mathbf{Set}} \longrightarrow \beta, \mu : \beta \circ \beta \longrightarrow \beta$, and (β, η, μ) is a monad.

(ii) (β, η, μ) is terminal among all monads over Set preserving finite coproducts with respect to monad morphisms.

2. Injectivity and Hausdorffness

Again let T denote an endofunctor of Set preserving finite coproducts. The purpose of this section is to find conditions, which are necessary and sufficient for the unique natural transformation $\lambda : T \longrightarrow \beta$ to be pointwise injective. If so, then for $1 := \{0\}$:

$$T1 = \emptyset \quad \text{or} \quad T1 \cong 1.$$

In the first case, $TX = \emptyset$ for every set X, because there is a map $T(X \longrightarrow 1) : TX \longrightarrow T1$. In the second case, there exists a unique natural transformation $\eta : id_{\mathbf{Set}} \longrightarrow T$. This follows immediately from Yoneda's Lemma, because $id_{\mathbf{Set}} \cong hom(1, -)$, the covariant hom-functor, hence

$$T1 \cong [id_{\mathbf{Set}}, T],$$

the set of all natural transformations from id_{Set} to T. Explicitly, η is given by

$$\{\eta_X(x)\} = T(\{x\} \hookrightarrow X)(T\{x\}), \quad x \in X.$$

By 1.2., η is pointwise dense with respect to the topology mentioned in 1.1. For the following assume $T1 \cong 1$.

2.1. Lemma. For any set Y and any continuous map $h : TY \longrightarrow T2$ there exists a unique map $g : Y \longrightarrow 2$ with $Tg = h$.

Proof. By assumption, $\eta_2 = id_2$ up to a natural isomorphism. Moreover, $T2$ is discrete and the following diagram commutes

$$\begin{tikzcd} Y \arrow[r, "g:=h\eta_Y"] \arrow[d, "\eta_Y"'] & 2 \arrow[d, equal, "\eta_2"] \\ TY \arrow[r, shift left, "h"] \arrow[r, shift right, "Tg"'] & T2 \end{tikzcd} .$$

This shows $h = Tg$, because η_Y is dense, as well as uniqueness of g.

□

2.2. Theorem. The following are equivalent (with respect to the topology mentioned in 1.1.):

(i) $\eta : T \longrightarrow \beta$ is pointwise injective.

(ii) For any map $f : X \longrightarrow TY$ and any source $(f_g : TX \longrightarrow T2)_{g \in 2^Y}$ such that $(Tf_g)\eta_X = (Tg)f$, there exists at most one diagonal d in

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & TX \arrow[d, "Tf_g"] \arrow[dl, dashed, "d"'] \\ TY \arrow[r, "Tg"'] & T2 \end{tikzcd},$$

i.e., $d\eta_X = f$ and $(Tg)d = Tf_g$ for all $g \in 2^Y$.

(iii) TY is Hausdorff (T_0) for every set Y.

(iv) The continuous maps $h : TY \longrightarrow T2$ form a mono–source for every set Y.

(v) $(Tg)_{g \in 2^Y}$ is a mono–source for every set Y.

Proof. Note, that Hausdorffness of TY follows from T_0, because there is a basis of clopen sets. This yields (iii) $\Rightarrow$ (iv), too. Moreover, (iv) $\Rightarrow$ (v) by 2.1., and (v) $\Rightarrow$ (i) $\Rightarrow$ (ii) are easy using uniqueness of d in case of $T = \beta$ and diagram chasing for the latter. For (v) $\Rightarrow$ (i) observe that the diagrams

$$\begin{tikzcd}
TY \arrow[r, "\lambda_Y"] \arrow[d, "Tg"'] & \beta Y \arrow[d, "\beta g"] \\
T2 \arrow[r, "\lambda_2"] \arrow[d, "\cong"'] & \beta 2 \arrow[d, "\cong"] \\
T1 \coprod T1 \arrow[r, "\lambda_1 \coprod \lambda_1", "\cong"'] & \beta 1 \coprod \beta 1
\end{tikzcd} \quad g \in 2^Y, \text{ commute,}$$

where $(Tg)_{g \in 2^Y}$ is a monosource. Therefore, λ_Y is injective.

For the remaining part (ii) $\Rightarrow$ (iii) assume that TY fails to be T_0 for some set Y, i.e. there are different points $s, t \in TY$ sharing the same neighbourhoods. By construction and $T1 \cong 1$, $\eta_Y(Y) \subseteq TY$ is a discrete open subspace, hence $s, t \in TY \setminus \eta_Y(Y)$. Moreover, $Tg(s) = Tg(t)$ for every $g \in 2^Y$, otherwise s, t could be sperated by a clopen set, because $T2 \cong 2$ discrete. This yields two diagonals in the following diagram

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[d, "\eta_Y"'] & TY \arrow[d, "Tg"] \arrow[dl, dashed, "id_{TY}, d"'] \\
TY \arrow[r, "Tg"'] & T2
\end{tikzcd} \quad g \in 2^Y,$$

$d(t) := s$, $d(u) := u$ otherwise.

□

2.3. Remark. Every diagonal d in the diagram 2.2.(ii) is continuous, because the topology on TY is initial with respect to $(Tg)_{g \in 2^Y}$. Obviously, conditions (i) and (iii) in 2.2. imply $T1 = \emptyset$ or $T1 \cong 1$. But this cannot be concluded from (v), hence neither (iv) nor (ii) (with respect to all natural transformations η). $T := id_{\mathbf{Set}} \coprod id_{\mathbf{Set}}$ serves as counterexample.

3. Surjectivity and Compactness

It would be nice, if existence of the diagonal d in 2.2.(ii) would imply pointwise surjectivity of $\lambda : T \longrightarrow \beta$. In fact, this turns out to be a necessary condition. It fails to be sufficient, as $T := id_{\mathbf{Set}}$ shows. But there is a wellknown property of β, shared by all T with pointwise surjective $\lambda : T \longrightarrow \beta$, which suffices together with the existence of d.

3.1. Lemma. Let X be a set of infinite cardinality card X. Then

$$(\beta(S \hookrightarrow X)(\beta S))_{S \subseteq X, cardS < cardX}$$

fails to cover βX.

Proof. If not, we have a cover of clopen sets, hence a finite subcover, thus even one subset $S \subseteq X$ with $cardS < cardX$ such that $\beta(S \hookrightarrow X)$ is surjective. This is impossible.

□

The following considerations enable us to restrict our attention again to the case $T1 \cong 1$.

3.2. Lemma. Let $\sigma : T \longrightarrow T^{\bullet}$ be a natural transformation between endofunctors of **Set** preserving finite coproducts and X a set, such that σ_X is surjective. Then

$$TX \text{ compact } \Leftrightarrow T^{\bullet}X \text{ compact}$$

(not necessarily Hausdorff, with respect to the topology introduced in 1.1.).

Proof. Obviously, σ_X induces a bijection between the bases of clopen sets in TX and $T^{\bullet}X$, respectively:

$$\sigma_X(T(S \hookrightarrow X)(TS)) = T^{\bullet}(S \hookrightarrow X)(T^{\bullet}S),$$
$$\sigma_X^{-1}(T^{\bullet}(S \hookrightarrow X)(T^{\bullet}S)) = T(S \hookrightarrow X)(TS).$$

Moreover, coverings consisting of such sets are carried over in coverings by σ_X and σ_X^{-1}, respectively.

□

3.3. Proposition. Let T be any endofunctor of **Set**. Then there exists a universal natural transformation $\sigma : T \longrightarrow T^{\bullet}$ to a functor $T^{\bullet} : \mathbf{Set} \longrightarrow \mathbf{Set}$ with $T^{\bullet}1 \cong 1$. Moreover, $T^{\bullet}$ preserves finite coproducts, if T does.

Proof. In case of $T1 = \emptyset$, $T^{\bullet} := id_{\mathbf{Set}}$ has the desired properties using the Yoneda lemma just as in the beginning of section 2. If $T1 \neq \emptyset$, take the (generalized) coequalizer of $[id_{\mathbf{Set}}, T]$ for $\sigma : T \longrightarrow T^{\bullet}$. Note $[id_{\mathbf{Set}}, T] \cong T1 \neq \emptyset$ by Yoneda's Lemma. Any natural transformation

$\nu : T \longrightarrow T'$ with $T'1 \cong 1$ coequalizes $[id_{\mathbf{Set}}, T]$, because $T'1 \cong [id_{\mathbf{Set}}, T']$. Hence, there exists a unique factorization

$$\begin{tikzcd} T \arrow[r, "\sigma"] \arrow[dr, "\nu"'] & T^{\bullet} \arrow[d, "\nu^{\bullet}"] \\ & T' \end{tikzcd}$$

Moreover, $T^{\bullet}$ inherits all colimit preservation properties from T.

□

By construction, $\lambda : T \longrightarrow \beta$ is pointwise surjective iff this holds for $\lambda^{\bullet} : T^{\bullet} \longrightarrow \beta$.

3.4. Theorem. Let T be an endofunctor of **Set** preserving finite coproducts with $T1 \cong 1$. Then the following are equivalent (with respect to the topology mentioned in 1.1.):

(i) $\lambda : T \longrightarrow \beta$ is pointwise surjective.

(ii) (a) Existence of the diagonal d with the same assumptions as in 2.2.(ii) and

(b) $(T(S \hookrightarrow X))_{S \subseteq X, cardS < cardX}$ fails to be an epi–sink for every set X with (regular) infinite cardinality.

(iii) TY is compact (not necessarily Hausdorff) for every set Y.

Proof. For simplicity assume $\lambda_2 = id_2$. For (a) consider the following commutative cube

$$\begin{tikzcd} & X \arrow[rr, "\beta_X"] \arrow[dd, "\lambda_Y f" near start] & & \beta X \arrow[dd, "\beta f_s"] \arrow[dl, dashed, "e"'] \\ X \arrow[ur, "id_X"] \arrow[rr, "\eta_X" near end] \arrow[dd, "f"'] & & TX \arrow[ur, "\lambda_X"'] \arrow[dd, "Tf_s" near start] \arrow[dl, dashed, "d"'] & \\ & \beta Y \arrow[rr, "\beta g" near start] & & \beta 2 \\ TY \arrow[ur, "\lambda_Y"] \arrow[rr, "Tg"'] & & T2 \arrow[ur, equal, "\lambda_2"'] & \end{tikzcd}$$

The diagonal e in the back exists according to universality of β_X. Commutativity of the top yields η_X to be injective, while λ_Y is surjective by (i). Thus, there exists a map $d : TX \longrightarrow TY$ such that

$$d(\eta_X(x)) = f(x), \qquad x \in X$$

$$\text{and} \quad d(t) \in \lambda_Y^{-1}(e(\lambda_X(t))) \;, \quad t \in TX \setminus \eta_X(X).$$

By construction, the upper triangle in the front commutes. The same holds for the lower triangle, because

$$Tg(d(\eta_X(x)) = Tg(f(x)) = Tf_g(\eta_X(x))$$

for $x \in X$ and for $t \in TX \setminus \eta_X(X)$

$$\begin{aligned} Tg(d(t)) &= \beta g(\lambda_Y(d(t))) = \beta g(e(\lambda_X(t))) \\ &= \beta f_g(\lambda_X(t)) \quad = Tf_g(t). \end{aligned}$$

Using 3.1., (b) is immediate from (i), thus (i) $\Rightarrow$ (ii).

Now assume (ii). To prove (iii), it suffices to show, that every ρ-sequence f in TY has a cluster point, where ρ runs through all infinite regular cardinals [2]. Therefore, consider the open ordinal space $[0, \rho[$ and the following diagrams, which do commute according to $\eta_2 = id_2$:

$$\begin{tikzcd} X = [0,\rho[\arrow[r, "\eta_X"] \arrow[d, "f"'] & TX \arrow[d, "T((Tg)f)"] \arrow[dl, dashed, "d"'] \\ TY \arrow[r, "Tg"'] & T2 \end{tikzcd} \qquad g \in 2^Y \quad .$$

By assumption (ii) (a), there exists a diagonal d, which is continuous (2.3.). Hence, if the ρ-sequence η_X has a cluster point in TX, then so does $f = d\eta_X$ in TY. For any ordinal $\alpha < \rho$ we have

$$TX = T([0,\alpha[\hookrightarrow X)(T[0,\alpha[) \coprod T([\alpha,\rho[\hookrightarrow X)(T[\alpha,\rho[),$$

$$\text{hence} \quad TX = (\bigcup_{\alpha<\rho} T([0,\alpha[\hookrightarrow X)(T[0,\alpha[)) \coprod (\bigcap_{\alpha<\rho} T([\alpha,\rho[\hookrightarrow X)(T[\alpha,\rho[)).$$

By (ii) (b), the first summand is different from TX, thus the second nonempty. Now

$$\begin{aligned} \overline{\eta_X([\alpha,\rho[)} &= \overline{T([\alpha,\rho[\hookrightarrow X)(\eta_{[\alpha,\rho[}([\alpha,\rho[))} \\ &\supseteq T([\alpha,\rho[\hookrightarrow X)(\overline{\eta_{[\alpha,\rho[}([\alpha,\rho[)}) \\ &= T([\alpha,\rho[\hookrightarrow X)(T[\alpha,\rho[), \end{aligned}$$

because $\eta_{[\alpha,\rho[}$ is dense, hence

$$\bigcap_{\alpha<\rho} \overline{\eta_X([\alpha,\rho[)} \neq \emptyset,$$

which means that η_X has a cluster point in TX.

(iii) $\Rightarrow$ (i) holds, because

$$\beta_X = \lambda_X \eta_X, \quad X \text{ a set},$$

is dense, hence $\lambda_X : TX \longrightarrow \beta X$ is dense, too, thus surjective by compactness of TX and Hausdorffness of βX.

□

4. Characterizations

Following the good old tradition to formulate axioms as weak as possible, we observe

4.1. Lemma. The following are equivalent for every endofunctor T of **Set** :

(i) T preserves finite coproducts.

(i)' T preserves *cosquares* $X \coprod X$.

Moreover, $T1 \cong 1 \Leftrightarrow T2 \cong 2$, if $T(1 \coprod 1) \cong T1 \coprod T1$.

Proof. The latter and (i) $\Rightarrow$ (i)' are obvious. Assume (i)'. Now $\emptyset \hookrightarrow 1 \coprod 1$ factorizes via both coproduct injections $i, j : 1 \hookrightarrow 1 \coprod 1$, hence $T(\emptyset \hookrightarrow 1 \coprod 1)$ via Ti and Tj, which are disjoint. Consequently, $T\emptyset = \emptyset$. It remains to show, that T preserves the coproduct X of two nonempty sets Y and Z:

$$\begin{tikzcd} \emptyset \neq Y \arrow[dr, hook, "u"'] & \\ & Y \coprod Z = X\,. \\ \emptyset \neq Z \arrow[ur, hook, "v"] & \end{tikzcd}$$

The injections are coretractions, i.e., $ru = id_Y$ and $sv = id_Z$ for some r and s. Consider the cosquare

$$X \overset{i}{\underset{j}{\rightrightarrows}} X \coprod X,$$

which is preserved by T. Then the following holds,

$$T(r \coprod s)Ti = T((r \coprod s)i) = TuTr,$$

and similar $T(r \coprod s)Tj = TvTs$. But $T(X \coprod X)$ is the (disjoint) union of the images of Ti and Tj, hence $T(Y \coprod Z)$ as image of $T(r \coprod s)$ is the union of the images of Tu and Tv. The latter are disjoint, because

$$T((Y \longrightarrow 1) \coprod (Z \longrightarrow 1))$$

separates them. Therefore,

$$\begin{tikzcd}
TY \arrow[dr, "Tu"] & \\
& T(Y \coprod Z) \\
TZ \arrow[ur, "Tv"'] &
\end{tikzcd}$$

is a coproduct.

□

The following is immediate from the preceding results.

4.2. Theorem. Let T be any endofunctor of Set. Then $T \cong \beta$ iff

(o) $T1 \cong 1$.

(i) T preserves cosquares.

(ii) For any map $f : X \longrightarrow TY$ and any source $(f_g : TY \longrightarrow T2)_{g \in 2^Y}$ such that the squares

$$\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[d, "f"'] & TX \arrow[d, "Tf_g"] \arrow[dl, dashed, "d"'] \\
TY \arrow[r, "Tg"'] & T2
\end{tikzcd} \quad g \in 2^Y$$

commute for every natural transformation $\eta : id_{\mathrm{Set}} \longrightarrow T$, there exists a unique diagonal d, rendering all the triangles commutative.

(iii) $(T(S \hookrightarrow X))_{S \subseteq X, card S < card X}$ fails to be an epi-sink for every set X with regular infinite cardinality.

These axioms (o)-(iii) are independent, as the following examples show (the proofs are left to the reader).

(o) Interpret $T := \beta \coprod \beta$ as monad of the category of compact Hausdorff spaces X together with one continuous, indempotent, unary operation $\omega : X \longrightarrow X$, $\omega\omega = \omega$.

(i) Let Y be a strongly-rigid compact Hausdorff space [6], $T\emptyset := \emptyset$, and

$$TX := Y^{(Y^X \setminus \{c : X \longrightarrow Y | c \text{ constant}\})}$$

for every nonempty set X. Now consider T as monad of the category of all powers Y^I together with $\emptyset$ [7,10].

(ii) For $T := \beta \coprod_{id_{\mathrm{Set}}} \beta$, the cokernelpair of $id_{\mathrm{Set}} \longrightarrow \beta$, the diagonal fails to be unique. Let α denote the first strongly compact cardinal [5]. If it exists, there is the following subfunctor T of the ultrafilter functor U :

$$TX := \{p \in UX \mid \|p\| < \alpha \text{ or } p\ \alpha\text{-complete}\}.$$

In this case, the diagonal does not exist in general.

(iii) Choose simply $T := id_{\mathbf{Set}}$.

4.3. Corollary (E.G. Manes [13, 14]). $\beta \cong U$, the ultrafilter functor.

Proof. Clearly, U fulfills (o) and (i). The diagonal is given by

$$d(p) := \{V \subseteq Y \mid U(V \hookrightarrow Y)(UV) \in f(p)\}, \quad p \in UX,$$

where $f(p)$ denotes the (ultra-)filter generated by $\{f(S) \mid S \in p\}$ in UY. Condition (iii) holds, because any ultrafilter refinement of

$$\{[\alpha, \rho[\mid \alpha < \rho\} \quad \text{in} \quad [0, \rho[\cong X$$

is not in the image of $U(S \hookrightarrow [0, \rho[)$ for $cardS < \rho$.

□

4.4. Corollary (R. Börger [4]). U is terminal among all endofunctors of Set preserving cosquares.

Recall, that a varietal theory $t : \mathbf{Set}^{op} \longrightarrow \mathbf{T}$ in the sense of F.E.J. Linton [12] has *rank* $\leq \rho$, $2 \leq \rho$ a regular cardinal, if each $\mathbf{T}$-morphism $\omega : tX \longrightarrow t1$ can be represented as a composition

$$\begin{tikzcd} tX \arrow[rr, "\omega"] \arrow[dr, "tf"'] & & t1 \\ & tS \arrow[ur, "\tau"'] & \end{tikzcd},$$

where $\tau : tS \longrightarrow t1$ is a $\mathbf{T}$-morphism and $cardS < \rho$. Obviously, it is possible to restrict to inclusions instead of arbitrary maps f [8].

Moreover, all $\mathbf{T}$-morphisms $\omega : tX \longrightarrow t1$ factorizing via $tf : tX \longrightarrow tS$ with $cardS \leq \rho$ generate a *subtheory* $t_{\leq \rho} : \mathbf{Set}^{op} \longrightarrow \mathbf{T}_{\leq \rho}$, i.e., $\mathbf{T}_{\leq \rho} \subseteq \mathbf{T}$ is a subcategory, $t_{\leq \rho}$ the corestriction of t, and a varietal theory, too.

4.5. Theorem. Let $t : \mathbf{Set}^{op} \longrightarrow \mathbf{T}$ be a varietal theory. Then t is the varietal theory of compact Hausdorff spaces (up to an isomorphism) iff

(o) There are only trivial binary operations, i.e., $hom(t2, t1) \cong 2$.

(i) $(t1)^2 \xrightarrow{(t(1\to Y))^2} (tY)^2 \xleftarrow{\Delta = \text{diagonal}} tY$ is an epi–sink in $\mathbf{T}$ for all sets Y.

(ii) For each set Y, $hom(t2, tY)$ is an epi–sink in $\mathbf{T}$.

(iii) The subtheories $t_{\leq\rho}$ fail to have rank $\leq \rho$ for every infinite regular cardinal ρ.

Proof. It is easy to check (o)–(iii) for $t = \beta^{op} : \mathbf{Set}^{op} \longrightarrow \mathbf{T}$, $\mathbf{T} = \beta^{op}(\mathbf{Set}^{op}) \subseteq \mathbf{Top}^{op}$, full, the varietal theory of compact Hausdorff spaces. Note, that the canonical maps

$$\beta 1 \coprod \beta 1 \longleftarrow \beta Y \coprod \beta Y \longrightarrow \beta Y$$

separate points. For the converse, it suffices to verify (o)–(iii) in 4.2. with respect to the corresponding monad

$$T = hom(-, t1) \circ t, \text{ i.e., } TX = hom(tX, t1).$$

By (o) above, $T2 \cong 2$. To prove (i) in 4.2., consider any cosquare

$$X \underset{j}{\overset{i}{\rightrightarrows}} X \coprod X$$

and the following commutative diagram

$$\begin{tikzcd}
 & T2 \prod TX \arrow[dl, "proj."'] \arrow[dr, "proj."] & \\
T2 & T(X \coprod X) \arrow[l, "Tc"'] \arrow[r, "T\nabla"] \arrow[u, "k"] & TX \\
 & TX \arrow[ul, shift left] \arrow[ul, shift right] \arrow[ul, phantom, "\neq"] \arrow[u, shift left, "Tj"'] \arrow[u, shift right, "Ti"] \arrow[ur, "idTX"'] &
\end{tikzcd}$$

where ∇ denotes the codiagonal and $c := (X \longrightarrow 1) \coprod (X \longrightarrow 1)$ ($t2 \cong (t1)^2$ and $t\nabla = \Delta!$). By assumption (i), Tc and $T\nabla$ form a mono–source. Hence, k is injective. By construction and using $T2 \cong 2$ we get

$$T2 \prod TX = (kTi)(TX) \cup (kTj)(X),$$

thus kTi and kTj form an epi–sink, forcing k to be surjective, too. Hence, T preserves cosquares, especially $1 \coprod 1$, and $T1 \cong 1$ by 4.1.

By (ii) above, $(Tg)_{g\in 2^Y}$ is a monosource, which guarantees uniqueness of the diagonal in 4.2.(ii). Existence follows from universality of η_X.

Condition (iii) above is a word for word translation of (iii) in 4.2.

□

4.6. Remark. A more detailed examination yields a much weaker, but ugly to state, version of (ii) above. Instead of Hausdorffness of the topology mentioned in 1.1., it suffices to assume, that the finer topology geneated by the images of all $hom(\omega, t1) : TS \longrightarrow TX$, X fixed, S any set, $\omega : tX \longrightarrow tS$ any T–morphism, as subbase of closed sets, is *weakly Soundararajan* [19,20]. This very weak separation condition implies Hausdorffness in the presence of T_1 and extemal disconnectedness. Any other separation axiom with the latter property would be sufficient.

Condition (iii) above cannot be replaced by the weaker statement, that t does not have a rank, at least if there exists some Ulam–measurable cardinal α [5]. In this case, the varietal theory t' corresponding to the ω^+–complete ultrafilter monad fails to have rank and fulfills (o)–(ii) (but not (iii), because $t'_{\leq\omega}$ has rank $\leq \omega$).

Recall, that algebraic categories in the sense of H. Herrlich [8,10] are equivalent to regular epireflective subcategories of their corresponding Eilenberg–Moore–categories. Moreover, the underlying set functor of **HComp** is well known to be monadic [12,13,14]. This yields an axiomatic description of **HComp** using the characterization of its varietal theory above. As in [9], call a subcategory of **Top** *nontrivial* if it contains a space with at least two points.

4.7. Corollary. A category **C** is equivalent to a nontrivial regular epireflective subcategory of **HComp** iff

HC 0: **C** has a terminal object 1, all copowers $1^{(I)}$ of it, and coequalizers.

HC 1: 1 is *projective* with respect to a **C**–morphism $s : X \longrightarrow Y$, i.e., every **C**–morphism $1 \to Y$ factorizes via s, iff s is a regular epimorphism.

HC 2: $1 \coprod 1 \xleftarrow{(X\to 1)\coprod(X\to 1)} X \coprod X \xrightarrow{\nabla = \text{ codiagonal}} X$ is a mono–source for each cosquare $X \coprod X$ in **C** and $hom(1, 1\coprod 1) \cong 2$.

HC 3: $1 \coprod 1$ is a coseparator in the full subcategory of all copowers of 1.

HC 4: For any (index–) set I with regular infinite cardinality there exists a **C**–morphism $1 \to 1^{(I)}$, which fails to factor via any $1^{(J\hookrightarrow I)}$ with $J \subseteq I$ and $card\ J < card\ I$.

Hence, maximality characterizes **HComp** up to equivalence:

HC 5: Any category $\mathbf{C}'$ fulfilling HC 0 - HC 4 and $\mathbf{C} \subseteq \mathbf{C}'$, full, is equivalent to $\mathbf{C}$,

or simply:

HC 5': Equivalence relations are effective in $\mathbf{C}$ [3].

Moreover, HC 0 - HC 2, HC 4 and

SHC 3: $1 \coprod 1$ is a coseparator in $\mathbf{C}$,

characterize **Stone**, the category of Stone spaces, and its dual **Boo**, the category of Boolean algebras [21].

Proof. Assume HC 0 - HC 4 for $\mathbf{C}$. By HC 0, $hom(1,-)$ has a left adjoint F and, together with HC 1, $hom(1,-)$ turns out to be algebraic. The corresponding varietal theory is just

$$F^{op} : \mathbf{Set}^{op} \longrightarrow F(\mathbf{Set})^{op},$$

the latter considered as full subcategory of $\mathbf{C}^{op}$. Now 4.5. applies, showing that F^{op} equals β^{op} up to an isomorphism of varietal theories. Hence the corresponding Eilenberg–Moore–category is (concretely) isomorphic to **HComp** and $\mathbf{C}$ is equivalent to a regular epireflective subcategory of **HComp**. HC 4 forces the latter to be nontrivial.

HComp fulfills HC 0 - HC 4. Consider any $\mathbf{C}'$ with HC 0 - HC 4 and $\mathbf{HComp} \subseteq \mathbf{C}'$. Then $\mathbf{C}'$ is equivalent to a regular epireflective, isomorphism closed subcategory $\mathbf{C}'' \subseteq \mathbf{HComp}$, and we have

$$\mathbf{HComp} \simeq \mathbf{C}^{\bullet} \subseteq \mathbf{C}'' \subseteq \mathbf{HComp},$$

where $\mathbf{C}^{\bullet} \subseteq \mathbf{C}''$ full and isomorphism closed. Using constant maps, any full embedding of a nontrivial subcategory in **Top** turns out to preserve terminality. Hence, one may assume $1 \in \mathbf{C}^{\bullet}$, thus by $hom_{\mathbf{C}^{\bullet}}(1,-) \cong hom_{\mathbf{HComp}}(1,-) \circ E$ for some equivalence $E : \mathbf{C}^{\bullet} \longrightarrow \mathbf{HComp}$ with $E1 = 1$,

$$hom_{\mathbf{C}^{\bullet}}(1,-) = hom_{\mathbf{HComp}}(1,-) \mid_{\mathbf{C}^{\bullet}}$$

is even varietal and natural isomorphic to the underlying set functor of $\mathbf{C}^{\bullet}$. Moreover, $hom_{\mathbf{C}^{\bullet}}(1, 1 \coprod 1)$ posesses exactly two elements, hence $1 \coprod 1 \cong 2 \in \mathbf{C}^{\bullet}$. Since $\mathbf{C}^{\bullet}$ is closed under the formation of limits in **HComp** (using uniqueness of the compact Hausdorff topology on a limit rendering all its projections continuous) all Stone spaces are contained in $\mathbf{C}^{\bullet}$ [18]. Consequently, $\beta(\mathbf{Set}) \subseteq$ *Stone* $\subseteq \mathbf{C}^{\bullet}$, $\beta \dashv hom_{\mathbf{C}^{\bullet}}(1,-)$, and $\mathbf{C}^{\bullet} = \mathbf{HComp}$ [17,18], hence $\mathbf{C}' \simeq \mathbf{HComp}$.

This shows, that **C** = **HComp** fulfills HC 5. Conversely, if **C** fulfills HC 0 – HC 4, then (up to equivalence) $\mathbf{C} \subseteq \mathbf{HComp}$, full, hence $\mathbf{C} \simeq \mathbf{HComp}$, if HC 5 holds. HC 5' forces **C** to be varietal [10,12], hence $\mathbf{C} \simeq \mathbf{HComp}$ as well [9].

□

4.8. Remark. HC 5 may be regarded as dual to Peano's induction axiom, which states minimality of $\mathbb{N}$. Note, that 4.7. constitutes a purely categorical characterization of **HComp** among all categories and not only among (certain) full subcategories of **Top** as in [9,15,17,18,20].

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