Category Theory at Work
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On the Existence and Structure of Free Topological Groups

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Abstract The subject indicated in the title serves as an example to explain the needs for and developements of certain categorical concepts. Categorical thinking is used first to find a more appropriate setting for attacking the problem (in looking at k-groups rather than topological groups); applying then techniques from categorical topology leads to a unified and conceptual treatment of the classical results in this field. Since the interrelation between algebra and topology in topological algebra is seen as of a basically categorical nature, particular emphasis is given to separate topological and algebraic arguments.

1 Introduction

1.1 Free Topological Groups and Category Theory

About half a century ago the Russian mathematician Markov was the first to prove the existence of a free (Hausdorff) topological group $G(X,\xi)$ over a given Tychonoff space (X,ξ), i.e. a continuous map (in fact an embedding) $\eta_{(X,\xi)}$ from (X,ξ) into (the underlying space of) a topological group $G(X,\xi)$ such that every continuous map f from (X,ξ) to (the underlying space of) any topological group A can be uniquely extended to a continuous homomorphism from $G(X,\xi)$ to A [11]. He constructed this object explicitely, but his construction was extremely complicated (more than 40 pages in print) and didn't really give much insight into the structure of free topological groups.

Hence very soon completely different proofs were given by Samuel [17] and Kakutani [5] independently; these proofs were not constructive at all but rather of a purely categorical nature. We will recall Samuel's proof in Section 2.1 for historical reasons since this seems to be the first appearance of an argument which later became known as *Freyd's General Adjoint Functor Theorem (GAFT)*. As is well known applying the GAFT does not yield any information about the internal structure of the universal objects; hence Samuel's proof, too, didn't show directly how the free topological group might look like algebraically or topologically.

A somewhat more instructive categorical approach in this respect is due to Wyler [21]; we will elaborate on this in section 2.2. One has to admit however, that the additional information inherent in Wyler's so called *Taut Lift Theorem* is getting lost as soon as seperation properties like Hausdorffness are taken into account. Nevertheless this shortcoming can be overcome in interesting situations as will be shown in section 3.

The main methodological difference between these two different categorical approaches can be best described as follows: Using the GAFT means using *abstract* category theory while Wyler's argument is a typical application of the theory of *concrete* categories.

In section 3 we investigate the structure of free topological groups to some extent. The method is categorical — or at least categorically minded: we first provide a suitable setup by replacing the

This work is an original contribution and will not appear elsewhere.

category **Top** of topological spaces by the category **k-Top** of k-spaces; due to *cartesian closedness* of the latter category a better categorical behaviour of the constructions involved is obtained which in turn allows a better insight into the topological structure of our universal objects without loosing the algebraic information given by the taut-lift approach.

1.2 Preliminaries

Throughout this paper we are concerned with the problem whether the functor V in the following commutative diagram of functors has a left adjoint G: **Top** $\to$ **TopGrp** and what this might look like.

$$\begin{array}{ccc} \mathbf{TopGrp} & \xrightarrow{V} & \mathbf{Top} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ \mathbf{Grp} & \xrightarrow[U]{} & \mathbf{Set} \end{array}$$

Here **Set** (respectively **Grp**, **Top**, **TopGrp**) denotes the category of sets (resp. groups, topological spaces, topological groups) and mappings (resp. homomorphisms, continuous maps, continuous homomorphisms), while the indicated functors are the obvious forgetful functors. By abuse of notation we will use the symbol V of the *underlying space functor* also for its possible (co-)restrictions, as e.g. in V: $\mathbf{Top_2Grp} \to \mathbf{Tych}$, where **Tych** denotes the full subcategory of **Top** of Tychonoff-spaces (we assume familiarity with the fact that a topological group, once it is Hausdorff, it automatically is Tychonoff; the reason for this is that every group topology is uniformizable (see e.g. [6])). The subscript 2 at **Top** as usual indicates Hausdorffness.

The following full subcategories of **Top** will play a particular role as base categories of V: **Creg**, the category of completely regular spaces (without T_1) and **FHaus**, the category of functionally Hausdorff spaces, where a space (X,ξ) is called *functionally Hausdorff* if any two points of (X,ξ) can be seperated by a continuous real-valued map. One obviuously has

$$\mathbf{Tych} \subset \mathbf{FHaus} \subset \mathbf{Top_2} \subset \mathbf{Top} \qquad \text{and} \qquad \mathbf{Tych} \subset \mathbf{Creg} \subset \mathbf{Top}$$

Each of these embeddings is reflective with continuous surjections as reflection maps: the first and the last one are even concretely reflective (see [4, 1.3.5] and [1, 5.22])

The category **Grp** is considered as a subcategory of the category **Γ-Alg** of associative Γ-algebras, where Γ is the group-type consisting of one nullary, one unary and one binary operation (see e.g. [10]). F: **Set** $\to$ **Grp** is the *free group functor* with $\eta_M: M \to UFM$ denoting the *insertion of generators map*; observe in particular that η_M factors over the free Γ-algebra ΓM over the set M (see [10, 1.2.5]). Given a map $f: M \to G$ from a set M into (the underlying set of) a group G, $f^\sharp$ will always denote its homomorphic extension to FM.

We would like to stress the following simple but crucial fact:

1.1 Lemma *Let* $(f_i: G \to (H_i, \tau_i))_{i \in I}$ *be a family of group homomorphisms into topological groups* (H_i, τ_i). *Then the initial topology* τ_{init} *on* G *with respect to this family is a group topology.*

Hence in particular the functor S: TopGrp $\to$ Grp *is a topological functor and the functor* V: TopGrp $\to$ Top *preserves initiality (i.e.* V *applied to an* S*-initial source yields a* T*-initial source).*

Proof Since the f_i's are homomorphisms, we have the commutative diagrams

$$\begin{array}{ccc} (G,\tau_{init}) & \xrightarrow{f_i} & (H_i,\tau_i) \\ i\downarrow & & \downarrow i \\ (G,\tau_{init}) & \xrightarrow[f_i]{} & (H_i,\tau_i) \end{array} \qquad \begin{array}{ccc} (G,\tau_{init})^2 & \xrightarrow{f_i^2} & (H_i,\tau_i)^2 \\ m\downarrow & & \downarrow m \\ (G,\tau_{init}) & \xrightarrow[f_i]{} & (H_i,\tau_i) \end{array}$$

where i (resp. m) denote the group-inversions (resp. multiplications). Now continuity of the operations on the groups (H_i,τ_i) together with initiality of the family of (continuous) maps f_i shows continuity of the group operations of (G,τ_{init}). ◇

As a simple consequence every subset A of a topological group (G,τ) *generates* a topological subgroup of (G,τ) in that one supplies the abstract subgroup of (the abstract) group G generated by A with the subspace topology.

The following notational conventions are used: a topological space is usually denoted by (X,ξ), where X is its underlying set and ξ its topology; similarly (G,τ) denotes a topological group with G its underlying group and τ its topology. We don't distinguish however notationally between a group and its underlying set (hence (G,τ) can also denote the underlying space of the topological group). Similarly we use the same symbol for morphisms in a concrete category and their underlying ones in the base category.

2 Existence Theorems

2.1 Abstract Methods - The GAFT

As mentioned in the introduction we will start recalling the first nearly categorical existence proof for free topological groups.

2.1 Proposition (Samuel) *The functor* V: **Top₂Grp** $\to$ **Top** *satisfies the conditions of the General Adjoint Functor Theorem and hence has a left adjoint.*

Proof Given a space (X,ξ) consider the class of all continuous maps $f:(X,\xi)\to VA$ from (X,ξ) into (the underlying space of) any topological group A. For every f the image $f[X]$ generates a topological subgroup A_f of the corresponding topological group A. By the following lemma 2.2 there is only a *set* $\mathcal{I}$ of pairwise non-isomorphic topological groups arising this way. Choose according to 2.2 for every f some isomorphism $\varphi_f: A_f\to I_f$ with $I_f\in\mathcal{I}$ and form the product $\prod_{I\in\mathcal{I}} I$ with projections π_I. There will result (keep in mind that V preserves products !) an induced continuous map Φ such that the following diagram commutes for every f:

$$\begin{array}{ccc} (X,\xi) & \xrightarrow{\Phi} & \prod_{I\in\mathcal{I}} I \\ f\downarrow & & \downarrow \pi_I \\ A_f & \xrightarrow[\varphi_f]{} & I_f \end{array}$$

It is now straightforward to check that the corestriction $\rho_{(X,\xi)}:(X,\xi)\to VG(X,\xi)$ of Φ to the (underlying space of the) topological subgroup of $\prod_{I\in\mathcal{I}} I$ generated by $\Phi[X]$ is V-universal for (X,ξ). ◇

2.2 Lemma *For every topological space (X, ξ) there exists only a* set $\mathcal{I}$ *of continuous maps* $i: (X, \xi) \to VI$ *with topological groups I such that for any continuous map $f: (X, \xi) \to VA$ there is some $i_f: (X, \xi) \to VI_f \in \mathcal{I}$ together with an isomorphism $\varphi: I_f \to A_f$ with $f = \varphi \circ i_f$.*

Proof The proof is standard: from a solution set for the set X with respect to U: Grp $\to$ Set (see e.g. [8]) one gets a solution set for the space (X, ξ) with respect to V by the observation that any set carries only a set of topologies. ◇

Remark Lemma 2.2 expresses — as mentioned in the proof — nothing else but the fact that Freyd's solution set condition is fulfilled with respect to the functor V. Hence the use of this crucial condition in the GAFT was already preceded in Samuel's argument. Certainly, Samuel — lacking the language of category theory — couldn't prove (or even state) the GAFT in full generality; however he in fact used the above argument for a quite general statement concerning (in today's language) the existence of left adjoints of (concrete) functors. Though the language of categories and functors was developed about the same time it took another dozen of years until the GAFT was established [3].

2.2 Concrete Methods - Taut Lifts

While the GAFT-approach focusses on a direct construction of a universal object the method discussed in this section starts asking whether it is possible to lift the adjunction at the bottom of our commutative diagram to get an adjunction at the top. We present the result of this approach in the following form, i.e. as an application of the first half of *Wyler's Taut-Lift-Theorem* [21] to our particular situation.

2.3 Theorem (Wyler) *Since S:* TopGrp $\to$ Grp *admits initial lifts and V:* TopGrp $\to$ Top *preserves initiality (see 1.1), for every topological space (X, ξ) there exists a V-universal arrow $\rho_{(X,\xi)}: (X, \xi) \to VG(X, \xi)$ which is a* lift *of the U-universal arrow $\eta_X: X \to UFX$ in the sense that $T\rho_{(X,\xi)} = \eta_X$ and $SG(X, \xi) = FX$.*
The topology of $G(X, \xi)$ is the initial (group-)topology on FX with respect to the family

$$\{f^\sharp: FX \to G \mid G = S(G, \tau),\ (G, \tau) \in \text{TopGrp},\ f: (X, \xi) \to S(G, \tau)\}.$$[1]

Proof Since S admits initial lifts, the initial group topology $\tau_{(X,\xi)}$ on FX described in the theorem exists; since V preserves initiality, it is also initial in Top with respect to the family of maps $f^\sharp$. Now everything can be read of the commutative triangles

$$\begin{array}{ccc} (X, \xi) & \xrightarrow{\eta_X} & (FX, \tau_{(X,\xi)}) \\ & \searrow^{f} & \downarrow f^\sharp \\ & & (G, \tau) \end{array}$$

◇

The following generalisation of 2.3 is easily obtained by first constructing the regular factorisation of the source of the $f^\sharp$'s and then taking the initial lift of the monosource of this factorisation. In this way the taut-lift approach is also applicable in situations where separation axioms are involved:

[1] loosely speaking this is the class of all homomorphic extensions of continuous maps from (X, ξ) to some topological group (G, τ)

2.4 Theorem ([2, 20]) *If in the commutative square of functors*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{V} & \mathbf{B} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ \mathbf{Grp} & \xrightarrow[U]{} & \mathbf{Set} \end{array}$$

S is a monotopological functor and V preserves initiality of monosources, then for every **B***-object B there exists a V-universal arrow $\rho_B: B \to VGB$ which is a lift of the composite $e \circ \eta_{TB}$ with some surjection e.*

Observe, that — as opposed to 2.3 — 2.4 no longer contains any information about the algebraic structure of the V−free objects, except one might be able to show that the source of the $f^{\cdot}$'s already is a monosource; this certainly will be the case if FX carries an **A**-structure (i.e. $FX = SA$) such that η_X is a **B**-morphism (i.e. $\eta_X = T(g:(X,\xi) \to VA)$), since then $\eta_X^{\cdot} = id_{FX}$. In this context the following result of Świrczkowski [19] is particularly helpful:

2.5 Proposition (Świrczkowski) *For every functionally Hausdorff space (X,ξ) the abstract free group FX over the underlying set of (X,ξ) carries a group-topology which is Tychonoff and makes η_X a continous map.*

We will give a proof of this result in section 3.2. Let us state here however the following immediate consequence of proposition 2.5 and the remarks preceding it:

2.6 Corollary *For every functionally Hausdorff space (X,ξ) the free Hausdorff topological group (= free topological group !) over (X,ξ) is algebraically the free group FX over the underlying set of (X,ξ).* ◇

One might ask whether even the free Hausdorff topological group over an arbitrary Hausdorff space (X,ξ) will be algebraically the free group over the underlying set of (X,ξ). Since adjunctions compose, this topological group is the free Hausdorff group over the **FHaus**-reflection $\chi(X,\xi)$ of (X,ξ), and hence algebraically the free group over the underlying set of $\chi(X,\xi)$. Since this reflection however fails to be one-to-one unless (X,ξ) is functionally Hausdorff itself, we can conclude:

2.7 Corollary *The free Hausdorff topological group over some space (X,ξ) is algebraically the free group $FX = FT(X,\xi)$ over the underlying set X of (X,ξ) iff (X,ξ) is functionally Hausdorff. Otherwise it is algebraically the free group $FT\chi(X,\xi)$ over the underlying set of the* **FHaus***-reflection $\chi(X,\xi)$ of (X,ξ).* ◇

3 The Structure of the Free Objects

While 2.7 gives us complete information about the algebraic structure of our free objects, 2.4 seems to give a reasonable description of the topology. A closer look however shows that this is not the case: the construction of the topology according to 2.4 requires knowledge of all the category $\mathbf{Top_2Grp}$ hence in particular of its free objects. It therefore hardly can contain a concrete description of the topology in terms of (X,ξ) as it would be desirable. Moreover 2.7 is based on 2.5, the original proof of which is extremely involved and not very instructive (see also [14]).

In this section we therefore will give a categorically minded approach to the study of the topological structure of free Hausdorff groups. Our starting point is the following obvious observation:

3.1 Lemma *Let the topological group (G,τ) be the free topological (resp. Hausdorff) group over the space (X,ξ) via the V-universal map $\rho_{(X,\xi)}$. Then τ is the largest (i.e. finest) group topology (resp. Hausdorff group topology) which allows $\rho_{(X,\xi)}$ to be continuous.* ◇

Given any topological group (G,τ) not only the group operations are continuous maps but also all maps from finite powers of (G,τ) into (G,τ) which can be derived from those, as e.g. $(x,y,z) \mapsto xy^{-1}z$ or $(x,y,z,u) \mapsto z^{-1}yx^2u$. Let us call these maps term maps (for a precise definition see any book on Universal Algebra) and let us denote by $\mathcal{T}_G$ the set of all term maps of a group G. If we want to describe the topology of the free topological group $G(X,\xi)$ over some space (X,ξ), 2.3 and 3.1 tell us that we have to look for a topology on FX as fine as possible such that the restrictions of all term maps from $\mathcal{T}_{FX}$ to the corresponding powers of the space (X,ξ) are continuous. This motivates the following definition due to Mal'cev [9].

3.2 Definition (Primitive Topology) Given a topological space (X,ξ) the final topology on FX with respect to the restrictions of the maps from $\mathcal{T}_{FX}$ to the corresponding topological powers of the space (X,ξ) is called the *primitive topology* on FX. It will be denoted by $\tau^p_{(X,\xi)}$.

Unfortunately — as already observed by Mal'cev — the primitive topology on FX in general fails to be a group topology; if it were, $(FX,\tau^p_{(X,\xi)})$ certainly would be the free topological group over (X,ξ) and we would have a reasonable description of the free topology at hand. The reason for this disappointing fact is a certain shortcoming of the category **Top**, as we will explain next.

3.1 The Topology of Free k-Groups

Instead of the category **Top** we will now work over the category **k-Top** of k-spaces. **k-Top** can most easily be defined as the mono-coreflective hull in **Top** of all compact Hausdorff spaces. Hence k-spaces are precisely the quotients of Hausdorff locally compact spaces. For us it will be enough to be familiar with the following important categorical properties of **k-Top**. Proofs for two of them are sketched by the references; for further details and a proof of the last mentioned property see [7].

- **k-Top** is a full concretely coreflective subcategory of **Top** (see [1, 16.5 (1)]. Hence in particular
- **k-Top** is a topological category whose final structures are final in **Top**, too (see [1, 21.30 – 21.35]).
- **k-Top** is cartesian closed.

As a bi-coreflective subcategory of **Top** the category **k-Top** has products; however, these *k-products* are (in general proper) refinements of the topological product of the spaces in question. In particular a pair (G,τ) of a group G and a k-topology τ on the underlying set of G such that the group operations are continuous with respect to the k-product will in general fail to be a topological group. We will call such an object a *k-group*. k-groups together with the continuous group homomorphisms form the category **k-Grp**. Though k-groups might fail to be topological groups we will see shortly that the study of free objects in **k-Grp** sheds considerable light on the topological structure of free topological groups. Observe first that for a k-space (X,ξ) there is an obvious notion of a *primitive k-structure* $\kappa^p_{(X,\xi)}$ on FX: in 3.2 you simply have to replace *topological power* by *k-power* (keep in mind that final structures in **k-Top** are final in **Top**,too).

Our first result shows that — due to cartesian closedness of **k-Top** (a crucial property in this context which the category **Top** is lacking) — the study of free k-groups is much easier than the study of free topological groups.

3.3 Theorem *For every k-space (X,ξ) the primitive k-structure is compatible, i.e. $(FX, \kappa^p_{(X,\xi)})$ is a k-group and hence the free k-group $G^k(X,\xi)$ over the k-space (X,ξ).*

Proof To prove continuity of the group inversion i simply observe that for every term map $t \in \mathcal{T}_{FX}$ the map $i \circ t$ again is a term map, and hence continuous with respect to $\kappa^p_{(X,\xi)}$; since this topology is final with respect to $\mathcal{T}_{FX}$ continuity of i follows. Similarly, continuity of the multiplication m follows from the observation that for every pair (t_1, t_2) of term maps the map $m \circ (t_1 \times t_2)$ is a term map again and that the family $\{t_1 \times t_2 \mid t_1, t_2 \in \mathcal{T}_{FX}\}$ is a final family by the following lemma. ◇

3.4 Lemma *Let* **C** *be a cartesian closed topological category and $(f_i: C_i \to C)_{i\in I}$ a final epi-sink in* **C**. *Then*

$$(f_i \times f_j: C_i \times C_j \to C \times C)_{i,j\in I}$$

is a final epi-sink again.

Proof Use [1, 27.22] and the fact that final epi-sinks are compositive. ◇

It will be useful to make some consequences of the previous theorem explicit. We here use the following notations:

- For a given set X the free monoid over the disjoint union of two copies of X is denoted by ΓX. $can_X: \Gamma X \to FX$ is the canonical representation of the free group over X as a (monoid) quotient of the free Γ-algebra ΓX over$X$. Remember that one might think of elements of ΓX as terms like e.g. $xy^{-1}yz$ with $x, v, z \in X$ where however $xy^{-1}yz$ is different from xz, but both will be identified by can_X. These terms again correspond to term maps; the number $a(t)$ of variables occuring in a term t will be called the *arity* of t.

- For a given k-space (X,ξ) the set ΓX can be (via its Γ-terms) given a k-topology in precisely the same way as in 3.2. The resulting space will be denoted by $\Gamma\widetilde{(X,\xi)}$. In complete analogy to 3.3 this is the free k-Γ-algebra over (X,ξ).

- By $\Gamma\widehat{(X,\xi)}$ we denote the space $\coprod_{t\in\Gamma X} (X,\xi)^{a(t)}$. Observe that there is a canonical surjection $quot_{(X,\xi)}: \coprod_{t\in\Gamma X} (X,\xi)^{a(t)} \to \Gamma\widetilde{(X,\xi)}$ characterized by $quot_{(X,\xi)} \circ \iota_t = t$ for every term map $t: (X,\xi)^{a(t)} \to \Gamma\widetilde{(X,\xi)}$ and with ι_t the corresponding coproduct injection.

- Assume any enumeration of (the restrictions of) the term maps of FX (!), starting with $t_0 = \eta_X$. We denote by F_kX the set $\bigcup_{l\leq k} t_l[X^{a(t_l)}]$. Observe that obviously $FX = \bigcup_{k=0}^{\infty} F_kX$. F_kX can be given the final topology with respect to the family $(t_l: (X,\xi)^{a(t_l)} \to F_kX)_{l\leq k}$. The resulting space will be denoted by $F_k\widetilde{(X,\xi)}$.

3.5 Corollary *For a k-space (X,ξ) the following hold:*

1. *The canonical representation is topologically a quotient map if considered as a map $can_{(X,\xi)}: \Gamma\widetilde{(X,\xi)} \to (FX, \kappa^p_{(X,\xi)})$.*

2. *The canonical surjection $quot_{(X,\xi)}: \Gamma\widehat{(X,\xi)} \to \Gamma\widetilde{(X,\xi)}$ is topologically a quotient map.*

3. *The k-space $(FX, \kappa^p_{(X,\xi)})$ is a colimit in* k-Top *of the chain of spaces $(\widetilde{F_kX})_{k\in\mathbb{N}}$.*

Proof 1. follows from the fact that FX endowed with the final topology with respect to $can_X: \Gamma(\widetilde{X,\xi}) \to FX$ is the free k-group over (X,ξ). This is a consequence of the observations that this final topology is compatible (use 3.4 and the fact that can_X is a Γ-Alg-morphism, i.e. consider diagrams similar to those in the proof of 1.1) and that the continuous homomorphic extension of any continuous map f from a k-space (X,ξ) into a k-group (G,τ) factors over $\Gamma(\widetilde{X,\xi})$ (use the corresponding fact for abstract algebras and finality of $can_{(X,\xi)}$).
2. is obvious since $quot_{(X,\xi)}$ is the first factor of a final sink (compare [1, 8.13]).
3. obviously is the case for the underlying sets; now use 3.3 and the fact that final sinks compose. ◇

We are now going to exploit the various desriptions of the topology of the free k-group $G^k(X,\xi)$ over a k-space (X,ξ). Our aim is here to study topological properties which $G^k(X,\xi)$ might inherit from (X,ξ). For doing so we introduce the following concepts:

3.6 Definition (t_2-spaces and k_ω-spaces) A k-space (X,ξ) is called

- *weakly Hausdorff* or a t_2-space provided the diagonal Δ_X is closed in the k-product $(X,\xi) \times (X,\xi)$. **k-Top$_2$** will denote the category of t_2-spaces.
- a k_ω-space, provided it is homeomorphic to a colimit of a countable chain of compact Hausdorff spaces. **k-Top$_\omega$** will denote the corresponding full subcategory of **k-Top**.

Since k-products are refinements of topological products every Hausdorff space is weakly Hausdorff. The most important feature of k_ω-spaces is the fact that their k-products and topological products coincide. This and some other topological properties we will make use of in the sequel are collected in the following proposition. For proofs we refer to the literature.

3.7 Proposition ([7, 12, 13, 15])

1. k-Top$_2$ *is a reflective subcategory of* k-Top; *its equalizers are precisely the closed embeddings.*
2. *A directed colimit of a chain of t_2-spaces is a t_2-space again.*
3. *If $q: (X,\xi) \to (Y,v)$ is a quotient in* k-Top *and (X,ξ) is t_2, then (Y,v) is t_2 if (and only if) $(q \times q)^{-1}[\Delta_Y]$ is closed in (X,ξ).*
4. *A t_2-space is k_ω, provided it is a quotient of some k_ω-space or a colimit of a countable chain of compact spaces.*
5. *Products of k_ω-spaces in* Top *and* k-Top *coincide.*
6. *k_ω-spaces are normal.* ◇

In order to prove that $G^k(X,\xi)$ inherits the t_2-property from the space (X,ξ) we will need the following criterion.

3.8 Proposition *For a weakly Hausdorff k-space (X,ξ) the free k-group $G^k(X,\xi)$ will be weakly Hausdorff again, provided for every pair (s,t) of term-maps of ΓX the set*

$$M(s,t) = \{(x,y) \in X^{a(s)} \times X^{a(t)} \mid can_X s(x) = can_X t(y)\}$$

is a closed subset of the k-product $(X,\xi)^{a(s)} \times (X,\xi)^{a(t)}$.

Proof In a first step we show that the space $\Gamma(\widehat{X,\xi})$ is t_2. By definition this space is the following countable directed colimit (with respect to a given enumeration of the term-maps of ΓX)

$$\Gamma(\widehat{X,\xi}) = \underset{k\to\infty}{\mathrm{colim}} \coprod_{l\leq k} (X,\xi)^{a(t_l)}.$$

Since the t_2-property is obviously stable under finite topological sums 3.7 (1.) and (2.) yield the desired result.
Using the abbreviation $q = can_X \circ quot_X$ we observe next the equality

$$M(s,t) = (q\times q)^{-1}[\Delta_{G^k(X,\xi)}] \cap (X^{a(s)} \times X^{a(t)}).$$

Now our hypothesis shows that $(q\times q)^{-1}[\Delta_{G^k(X,\xi)}]$ has a closed intersection with every summand of

$$\Gamma(\widehat{X,\xi}) \times \Gamma(\widehat{X,\xi}) = \coprod_{(s,t)} (X,\xi)^{a(s)} \times (X,\xi)^{a(t)}$$

and hence is closed in $\Gamma(\widehat{X,\xi})^2$. Now apply 3.7 (3.) and 3.5. ◇

Applying our criterion 3.8 we will get the desired result. Instead of presenting a complete proof. which would be quite technical, we will illustrate the power of the criterion by a typical example.

Consider the following terms from ΓX : $s = xy^{-1}r^{-1}rzw$ and $t = uw$. The (restrictions of the) corresponding term-maps are the maps

$$\begin{array}{llll} s\colon X^5 & \longrightarrow & FX \\ (a_1,\dots,a_5) & \mapsto & a_1a_2^{-1}a_3^{-1}a_3a_4a_5 \end{array} \quad\text{and}\quad \begin{array}{lll} t\colon X^2 & \longrightarrow & FX. \\ (b_1,b_2) & \mapsto & b_1b_2 \end{array}$$

It is easy to compute the set $M(s,t)$ as $M(s,t) = M_1 \cup M_2$ with

$$M_1 = \{(a_1,a_1,a_3,a_4,a_5,a_4,a_5) \mid a_i \in X\} \quad\text{and}\quad M_2 = \{(a_1,a_2,a_3,a_2,a_5,a_1,a_5) \mid a_i \in X\}.$$

Since $M_1 = X^3 \times \Delta_{X^2}$ and $M_2 \simeq X \times \Delta_X^3$, the set $M(s,t)$ is a union of two closed subsets of $(X,\xi)^7 = (X,\xi)^{a(s)} \times (X,\xi)^{a(t)}$ and therefore closed.

Also in the general case (for a complete proof see [16]) $M(s,t)$ will be a finite union of subsets which are homeomorphic to subspaces of $(X,\xi)^{a(s)} \times (X,\xi)^{a(t)}$ of the form $\Delta_X^k \times X^l$ with $2k+l = a(s)+a(t)$. So we have the following result.

3.9 Theorem (Lamartin) *For every weakly Hausdorff k-space (X,ξ) the free k-group $G^k(X,\xi)$ is weakly Hausdorff again.* ◇

It is not difficult now to prove, that $G^k(X,\xi)$ also inherits the k_ω-property from (X,ξ).

3.10 Theorem *For every k_ω-space (X,ξ) the free k-group $G^k(X,\xi)$ is k_ω again and contains (X,ξ) as a closed subspace.*

Proof In a first step let us assume that (X,ξ) is even compact Hausdorff. Then $\Gamma(\widehat{X,\xi})$ is a k_ω-space by the representation given in the proof of 3.8, since finite sums of powers of compact Hausdorff spaces are compact Hausdorff. By 3.9 and 3.7 (4.) we are done. Alternatively: because of 3.9 the representation given in 3.5 (3.) fulfills the second condition of 3.7 (4.).
Let now (X,ξ) be given as a colimit of the chain $((X_n,\xi_n))_{n\in\mathbb{N}}$ of compact Hausdorff spaces. For every $n \in \mathbb{N}$ there is by the first step of this proof a representation of (the underlying space of) $G^k(X,\xi)_n$ as a colimit of a chain of compact Hausdorff spaces $(C_k^n)_{k\in\mathbb{N}}$. But then

$G^k(X,\xi)$ is, as a space, a colimit of the chain $(C_n^n)_{n\in N}$ and therefore a k_ω-space. This argument uses the facts that (i) G^k as a left adjoint and (ii) the underlying space functor V into the cartesian closed topological category **k-Top** (by [15, 3.1]) preserve directed colimits, such that $VG^k(X,\xi) = \mathrm{colim}_{n\to\infty} VG^k(X,\xi)_n$, and that there is (iii) a particular version of the theorem on limits with parameters (see [8]) valid for topological categories (see [15, 4.1]).
Starting from 3.5 (3.) the same procedure shows closedness of (X,ξ) in $VG(X,\xi)$. ◇

3.2 The Topology of Free Topological Groups

We now will exploit the results of the previous section to get additional information about the topology of the free *topological* group $G(X,\xi)$ over a topological space (X,ξ). The key for doing this is the following proposition.

3.11 Proposition *For every k_ω-space (X,ξ) the free topological group $G(X,\xi)$ coincides with the free k-group $G^k(X,\xi)$. In particular, the primitive topology is compatible, and $G(X,\xi)$ is simply $(FX,\tau^p_{(X,\xi)})$.*

Proof Since $G^k(X,\xi)$ is a k_ω-space by 3.10 it follows from 3.7 (5.) that $G^k(X,\xi)$ also is a topological group. Moreover 3.7 (5.) implies — in connection with the fact that final structures in **k-Top** are final in **Top**, too —, that the primitive k-structure $\kappa^p_{(X,\xi)}$ is the primitive topology $\tau^p_{(X,\xi)}$. ◇

The following is then an immediate consequence of this observation.

3.12 Theorem (Mal'cev) *For every k_ω-space (X,ξ) the free topological group $G(X,\xi)$ is a normal space and contains (X,ξ) as a closed subspace.*

Proof Use 3.11 and 3.10 in connection with 3.7 (6.). ◇

By a different but also very simple method we can generalize from compact Hausdorff spaces to Tychonoff spaces. Here we obtain first the main part of Świrczkowski's result 2.5 where our proof in addition gives a conceptual description of the constructed topology. We should add however, that we don't know whether both constructions coincide.

3.13 Proposition *For every Tychonoff space (X,ξ) the free abstract group FX carries a compatible Tychonoff topology $\sigma_{(X,\xi)}$. The topological group $(FX,\sigma_{(X,\xi)})$ is the free completely regular group over (X,ξ), which happens to be Tychonoff and hence also is the free Tychonoff group*[2] *over (X,ξ) and contains the space (X,ξ) as a (closed*[3]*) subspace.*

Proof Replace in the proof of 2.3 the category **Top** by the category **Creg** (which also admits arbitrary initial structures as a concretely reflective subcategory of **Top** (see [1, 21.35])) and obtain a topological group $G^c(X,\xi) = (FX,\sigma_{(X,\xi)})$ which is free over the space (X,ξ) in the category **CregGrp** of completely regular groups via the insertion of generators map η_X. Let now $\beta\colon(X,\xi)\to\beta(X,\xi)$ be the Stone-Čech compactification of (X,ξ). Since $G\beta(X,\xi)$ belongs to **CregGrp** by 3.12 the universal property of $G^c(X,\xi)$ yields a continuous injection $F\beta\colon G^c(X,\xi)\to G\beta(X,\xi)$ such that the following diagram commutes.

$$\begin{array}{ccc} (X,\xi) & \xrightarrow{\eta_{(X,\xi)}} & G^c(X,\xi) \\ \beta\big\downarrow & & \big\downarrow F\beta \\ \beta(X,\xi) & \xrightarrow[\eta_{\beta(X,\xi)}]{} & G\beta(X,\xi) \end{array}$$

[2]We rather should have said *Hausdorff* group here; for the reason of our terminology see the final remark.
[3]We won't discuss closedness of (X,ξ) here, see e.g. [18]

Since $G\beta(X,\xi)$ is Hausdorff by 3.12, it follows that $\sigma_{(X,\xi)}$ must be Hausdorff, too. Hence $\sigma_{(X,\xi)}$ is a Tychonoff topology. The diagram shows in addition that $\eta_{(X,\xi)}$ must be an embedding since the embedding $\eta_{\beta(X,\xi)} \circ \beta$ factors over $\eta_{(X,\xi)}$. ◇

Remark Świrczkowski's complete result (see 2.5) is now a simple corollary. If (X,ξ) is a functionally Hausdorff space, consider first its **Tych**-reflection $\rho_{(X,\xi)}: (X,\xi) \to \rho(X,\xi)$, which as a map is the identity id_X. Hence $FT(X,\xi) = FT\rho(X,\xi)$, and 3.13 gives the result. Note that — though $\eta_{(X,\xi)}$ and $\eta_{\rho(X,\xi)}$ coincide as maps — $\eta_{\rho(X,\xi)}$ no longer is an embedding, except (X,ξ) already was a Tychonoff space.

Let us finally collect some further conclusions as our last theorem.

3.14 Theorem *Let (X,ξ) be a topological space and $G(X,\xi)$ the free topological group over (X,ξ). Then $G(X,\xi) = (FX,\tau_f)$ and the universal continuous map $\eta_{(X,\xi)}: (X,\xi) \to G(X,\xi)$ is as a map simply the insertion of generators map $\eta_X: X \to FX$. Moreover the following hold*

1. *(X,ξ) is Tychonoff $\iff$ $G(X,\xi)$ is Tychonoff*[4] *and $\eta_{(X,\xi)}$ is an embedding*
2. *(X,ξ) is functionally Hausdorff $\iff$ $G(X,\xi)$ is functionally Hausdorff*[4].

In both cases $G(X,\xi)$ is also the free Hausdorff group.

If however (X,ξ) is not functionally Hausdorff then the free Hausdorff group over (X,ξ) is the free topological group $G\chi(X,\xi)$ over the FHaus*-reflection $\chi(X,\xi)$ of (X,ξ).*

Proof If (X,ξ) is a functionally Hausdorff space, then $id_{FX}: G(X,\xi) \to (FX, \sigma_{(X,\xi)})$ is the continuous (!) homomorphic extension of $\eta_{(X,\xi)}: (X,\xi) \to (FX,\sigma_{(X,\xi)})$; it follows that the topology of $G(X,\xi)$ refines the Tychonoff topology $\sigma_{(X,\xi)}$ and hence is functionally Hausdorff. A (functionally) Hausdorff group however is Tychonoff automatically.
If (X,ξ) is even Tychonoff, $\eta_{(X,\xi)}: (X,\xi) \to G(X,\xi)$ is an embedding as can be seen using the same argument as in the proof of 3.13; one only has to replace $G^c(X,\xi)$ by $G(X,\xi)$.
The remaining statements are obvious. ◇

Remark Throughout this note *group* can be replaced by *equationally defined algebra with respect to a fixed finitary typ* Ω. If then Ω is used instead of the group-type Γ and the corresponding variety instead of Grp, all results hold without any change in the proofs, except for statement (1.) in 3.14 which would only read as follows

- *(X,ξ) is Tychonoff $\Longrightarrow$ $G(X,\xi)$ is functionally Hausdorff and $\eta_{(X,\xi)}$ is an embedding*

(the proof of 3.14 here made use of the fact, that a functionally Hausdorff group is Tychonoff, and hence can only be extended to algebras like abelian groups or rings).

It is still an open question whether 3.14 (1.) holds for algebras in general, i.e. whether the free topological algebra over a Tychonoff space is Tychonoff, even if the algebra has no underlying group structure.

References

[1] Adámek, J. Herrlich, H., and G.E. Strecker, *Abstract and Concrete Categories*, Wiley Interscience, New York, 1990

[2] Fay, T.H., An Axiomatic Approach to Categories of Topological Algebras, *Quaestiones Math.* 2 (1977), 113–137

[3] Freyd, P., *Functor Theory*, PhD Dissertation, Princeton University, Princeton, 1960

[4]see footnote 2

[4] Herrlich, H., *Topologische Reflexionen und Coreflexionen* , Lecture Notes in Mathematics **78**, Springer, Berlin, 1968

[5] Kakutani, S., Free Topological Groups and Finite Discrete Product Topological Groups, *Proc. Imp. Acad. Tokyo* **20** (1944), 595–598

[6] Kelley, J.L., *General Topology*, GTM **27**, Springer, New York

[7] Lamartin, W.F., *On the Foundations of k-group Theory*. Dissertationes Math. **CXLVI**, Warsaw, 1977

[8] Mac Lane, S., *Categories for the Working Mathematician*, GTM **5**, Springer, New York, 1971

[9] Mal'cev, A.I., Free Topological Algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* **21** (1957), 171–198, [*Amer. Math. Soc. Transl. Ser. 2* **17** (1961), 173–200]

[10] Manes, E.G., *Algebraic Theories*, GTM **26**, Springer, New York, 1976

[11] Markov, A.A., On Free Topological Groups, *Izv. Akad. Nauk. SSSR Ser. Mat.* **9** (1945), 3–64, [*Amer. Math. Soc. Transl.* **30** (1950), 11–88; Reprint: *Amer. Math. Soc. Transl. Ser. 1* **8** (1962), 195–272]

[12] McCord, M.C., Classifying Spaces and Infinite Symmetric Products, *Trans. Amer. Math. Soc.* **146** (1969), 273–298

[13] Ordman, E.T., Free *k*-groups and Free Topological Groups, *(General) Topology Appl.* **5** (1975), 205–219

[14] Porst, H.-E., On Free Topological Algebras, *Cahiers Topologie Géom. Différentielles Catégoriques* **28** (1987), 235–253

[15] Porst, H.-E., Free Algebras over Cartesian Closed Topological Categories, in: Frolík, Z. (ed.), *General Topology and its Relations to Modern Analysis and Algebra VI, Proc. Sixth Prague Topological Symposium 1986*, Heldermann Verlag, Berlin, 1988, 437 - 450

[16] Porst, H.-E., Separation in Free Algebras over *Top* and *k-Top*, in: Adámek, J. and S. MacLane (eds.), *Categorical Topology and its Relations to Analysis, Algebra and Combinatorics, Proc. Conf. Prague 1988*, World Scientific, Singapore, 1989, 238 - 245

[17] Samuel, P., On Universal Mappings and Free Topological Groups, *Bull. Amer. Math. Soc.* **54** (1948), 591–598

[18] Smith-Thomas, B., Free Topological Groups, *(General) Topology Appl.* **4** (1974), 51–72

[19] Świrczkowski, S., Topologies in Free Algebras, *Proc. London Math. Soc. (3)* **14** (1964), 566–576

[20] Tholen, W., On Wyler's Taut Lift Theorem, *(General) Topology Appl.* **8** (1978), 197–206

[21] Wyler, O., On the Categories of General Topology and Topological Algebra, *Arch. Math.* **XXII** (1971), 7–17