Category Theory at Work
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Point Separation Axioms, Monotopological Categories and MacNeille Completions

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Abstract. Using results of ALDERTON, HERRLICH, PREUß, and SCHWARZ the interrelations between classes of objects (in a topological category) defined by "point separation axioms", monotopological categories and MACNEILLE completions are clarified.

§0 Introduction

Generalizing quasicomponents introduced by HAUSDORFF [3] to $\mathcal{P}$-quasicomponents (cf. PREUß [8]) for each class $\mathcal{P}$ of topological spaces one obtains important classes of topological spaces defined by point separation axioms as classes $Q\mathcal{P}$ of totally $\mathcal{P}$-separated spaces, i.e. spaces whose $\mathcal{P}$-quasicomponents are singletons, for suitable $\mathcal{P}$'s. Replacing the category Top of topological spaces (and continuous maps) by an arbitrary topological category $\mathcal{C}$ in the sense of HERRLICH [5] and introducing classes $Q\mathcal{P}$ of totally $\mathcal{P}$-separated objects of $\mathcal{C}$ for each class $\mathcal{P}$ of $\mathcal{C}$-objects the problem of characterizing these classes arises. It turns out that they are identical with the object classes of the extremal epireflective subcategories of $\mathcal{C}$.

NEL [7] has introduced initially structured categories, which are now also called monotopological categories, as a generalization of topological categories in order to exclude no longer such an important category as the category Haus of HAUSDORFF spaces (and continuous maps) from consideration. Using results of HERRLICH [4], SCHWARZ [10] has shown that a concrete category is monotopological iff it is an extremal epireflective subcategory of some topological category.

The classical construction of the MACNEILLE completion of a poset was generalized by HERRLICH [6] to the construction of the MACNEILLE completion of a concrete category. Since MACNEILLE completions of concrete categories need not always exist, it is remarkable that a monotopological category $\mathcal{C}$ may be characterized as a (concrete) category which is an extremal epireflective subcategory (up to concrete isomorphim) of a topological category being a MACNEILLE completion of $\mathcal{C}$ (cf. ALDERTON [1]).

Thus, a concrete category $\mathcal{C}$ is monotopological iff its object class is a class $Q\mathcal{P}$ of totally $\mathcal{P}$-separated objects with respect to a topological category which is a MACNEILLE completion of $\mathcal{C}$.

This work is an original contribution and will not appear elsewhere.

§1 Preliminaries

1.1. Let $\mathcal{X}$ be a fixed category, called *base category*. A *concrete category* over $\mathcal{X}$ is a pair $(\mathcal{A}, \mathcal{F})$, where $\mathcal{A}$ is a category and $\mathcal{F}: \mathcal{A} \to \mathcal{X}$ a functor which is

(1) *faithful*,

(2) *amnestic*, i.e. any $\mathcal{A}$-isomorphism f is an $\mathcal{A}$-identity iff $\mathcal{F}(f)$ is an $\mathcal{X}$-identity,

and

(3) *transportable*, i.e. for each $\mathcal{A}$-object A, each $\mathcal{X}$-object B and each isomorphism $q: B \to \mathcal{F}(A)$, there exists a unique $\mathcal{A}$-object C and an isomorphism $\overline{q}: C \to A$ with $\mathcal{F}(\overline{q}) = q$,

called the *underlying functor* of $(\mathcal{A}, \mathcal{F})$. Occasionally, $(\mathcal{A}, \mathcal{F})$ is denoted by $\mathcal{A}$ too.

1.2 Definitions. [1] Let $\mathcal{F}: \mathcal{A} \to \mathcal{X}$ be a functor. A pair $\left(A, (f_i: A \to A_i)_{i \in I}\right)$, where A is an $\mathcal{A}$-object and $(f_i: A \to A_i)_{i \in I}$ a class-indexed family of $\mathcal{A}$-morphisms each with domain A, called a *source* in $\mathcal{A}$, is $\mathcal{F}$*-initial* iff for each source $\left(B, (g_i: B \to A_i)_{i \in I}\right)$ in $\mathcal{A}$ and each $\mathcal{X}$-morphism $f: \mathcal{F}(B) \to \mathcal{F}(A)$ such that $\mathcal{F}(f_i) \circ f = \mathcal{F}(g_i)$ for each $i \in I$, there exists a unique $\mathcal{A}$-morphism $\overline{f}: B \to A$ with $\mathcal{F}\left(\overline{f}\right) = f$ and $f_i \circ \overline{f} = g_i$ for each $i \in I$. Dually: *sink*, $\mathcal{F}$*-final*.

[2] A functor $\mathcal{F}: \mathcal{A} \to \mathcal{X}$ is called *topological* (resp. *monotopological*) provided that for each class-indexed family $(A_i)_{i \in I}$ of $\mathcal{A}$-objects and each source $\left(X, (f_i: X \to \mathcal{F}(A_i))_{i \in I}\right)$ in $\mathcal{X}$ (resp. each mono-source $\left(X, (f_i: X \to \mathcal{F}(A_i))_{i \in I}\right)$ in $\mathcal{X}$, i.e. each source $\left(X, (f_i: X \to \mathcal{F}(A_i))_{i \in I}\right)$ in $\mathcal{X}$ such that for any pair $Y \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X$ of $\mathcal{X}$-morphisms with $f_i \circ \alpha = f_i \circ \beta$ for each $i \in I$ it follows that $\alpha = \beta$) there exists a unique $\mathcal{F}$-initial source $\left(A, (g_i: A \to A_i)_{i \in I}\right)$ in $\mathcal{A}$ with $\mathcal{F}(A) = X$ and $\mathcal{F}(g_i) = f_i$ for each $i \in I$.

[3] A concrete category $(\mathcal{A}, \mathcal{F})$ over the category **Set** of sets (and maps) is called *topological* (resp. *monotopological*) provided that the following are satisfied:

(a) $\mathcal{F}: \mathcal{A} \to$ **Set** is topological (resp. monotopological).

(b) $\mathcal{F}: \mathcal{A} \to$ **Set** has small fibres, i.e. for such $X \in |\,\mathbf{Set}\,|, \{A \in |\,\mathcal{A}\,| : \mathcal{F}(A) = X\}$ is a set.

(c) There is precisely one object P in $\mathcal{A}$ (up to isomorphism) such that $\mathcal{F}(P)$ is a singleton (= terminal separator).

1.3 Remarks. [1] Let $(\mathcal{A}, \mathcal{F})$ be a concrete category over $\mathcal{X}$. Then $\mathcal{F}: \mathcal{A} \to \mathcal{X}$ is topological iff it is *cotopological*, i.e. for each class-indexed family $(A_i)_{i \in I}$ of $\mathcal{A}$-objects and each sink $\left((f_i: \mathcal{F}(A_i) \to X)_{i \in I}, X\right)$ in $\mathcal{X}$ there exists a unique $\mathcal{F}$-final sink $\left((g_i: A_i \to A)_{i \in I}, A\right)$ in $\mathcal{A}$ with $\mathcal{F}(A) = X$ and $\mathcal{F}(g_i) = f_i$ for each $i \in I$.

[2] A concrete category $\mathcal{A}$ over **Set** may be considered to be a category $\mathcal{A}$ whose objects are structured sets, i.e. pairs (X, ξ) where X is a set and ξ is an $\mathcal{A}$-structure on X, whose morphism $f: (X, \xi) \to (Y, \eta)$ are suitable maps between X and Y and whose composition law is the usual composition of maps. Thus, (a) may be replaced by

(a′) For any set X, any family $((X_i,\xi_i))_{i\in I}$ of $\mathcal{A}$-objects indexed by a class I and any family $(f_i : X \to X_i)_{i\in I}$ (resp. any family $(f_i : X \to X_i)_{i\in I}$ such that for each pair $(x,y) \in X \times X$ with $x \neq y$ there exists some $i \in I$ with $f_i(x) \neq f_i(y)$) indexed by I there exists a unique $\mathcal{A}$-structure ξ on X which is *initial* with respect to $(X, f_i, (X_i, \xi_i), I)$, i.e. such that for any $\mathcal{A}$-object (Y,η) a map $g : (Y,\eta) \to (X,\xi)$ is an $\mathcal{A}$-morphism iff for every $i \in I$ the composite map $f_i \circ g : (Y,\eta) \to (X_i,\xi_i)$ is an $\mathcal{A}$-morphism.

[3] Monotopological categories are also called *initially structured categories* (cf. [9]).

[4] Each monotopological category is topological. The converse is not true (cf. the following examples).

1.4 Examples. The categories **Top**, **Unif**, **Prox**, **Near**, **Lim** and **Prost** consisting of topological spaces, uniform spaces, proximity spaces, nearness spaces, limit spaces, and preordered sets respectively are topological categories whereas the categories **Haus** and **Poset** consisting of HAUSDORFF spaces and partially ordered sets respectively are monotopological but not topological. In each case the underlying functor is the usual forgetful functor into **Set**. For further details see [9].

1.5. In the following subcategories of a category $\mathcal{C}$ are always assumed to be *full and isomorphism-closed* (a subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is isomorphism-closed iff each $\mathcal{C}$-object being isomorphic to some $\mathcal{A}$-object is an $\mathcal{A}$-object).

Definition: If $\mathcal{A}$ is a subcategory of some category $\mathcal{C}$ and $\mathcal{I} : \mathcal{A} \to \mathcal{C}$ denotes the inclusion functor, then $\mathcal{A}$ is called *reflective* (resp. *coreflective*) in $\mathcal{C}$ provided that one of the following (equivalent) conditions is satisfied:

(1) $\mathcal{I}$ has a left-adjoint R (resp. right-adjoint R_c) called a *reflector* (resp. a *coreflector*).

(2) For each $\mathcal{C}$-object X, there exist an $\mathcal{A}$-object $X_{\mathcal{A}}$ and an $\mathcal{C}$-morphism $r_X : X \to X_{\mathcal{A}}$ called an $\mathcal{A}$*-reflection* of X (resp. $m_X : X_{\mathcal{A}} \to X$ called an $\mathcal{A}$*-coreflection* of X) such that for each $\mathcal{A}$-object Y and each $\mathcal{C}$-morphism $f : X \to Y$ (resp. $f : Y \to X$) there is a unique $\mathcal{A}$-morphism (= $\mathcal{C}$-morphism) $\overline{f} : X_{\mathcal{A}} \to Y$ (resp. $\overline{f} : Y \to X_{\mathcal{A}}$) such that $\overline{f} \circ r_X = f$ (resp. $m_X \circ \overline{f} = f$). Further, $\mathcal{A}$ is called *epireflective* (*monocoreflective*), *extremal epireflective* (*extremal monocoreflective*) or *bireflective* (*bicoreflective*) in $\mathcal{C}$ respectively provided that $\mathcal{A}$ is reflective (coreflective) in $\mathcal{C}$ and for each $\mathcal{C}$-object X, the $\mathcal{A}$-reflections ($\mathcal{A}$-coreflections) of X are epimorphisms (monomorphisms), extremal epimorphisms (extremal monomorphisms) or bimorphisms respectively.

1.6 Proposition (cf. [9; 2.2.12] for the proof). Any bireflective (and any bicoreflective) subcategory of a topological category is a topological category.

1.7 Remark. Similar to the situation for the category **Top** we obtain that in topological categories extremal monomorphisms coincide with embeddings and extremal epimorphisms coincide with quotient maps. Furthermore, the following proposition is valid:

1.8 Proposition. Let $\mathcal{A}$ be a subcategory of a topological category $\mathcal{C}$. Then the following hold:

(1) $\mathcal{A}$ is epireflective (extremal epireflective) in $\mathcal{C}$ iff $\mathcal{A}$ is closed under formation of products and subobjects [= extremal monomorphisms] (weak subobjects [= monomorphisms]) in $\mathcal{C}$.

(2) If $\mathcal{A}$ contains at least one object with non-empty underlying set, then the following are equivalent:

  (a) $\mathcal{A}$ is coreflective in $\mathcal{C}$.

  (b) $\mathcal{A}$ is bicoreflective in $\mathcal{C}$.

  (c) $\mathcal{A}$ is closed under formation of coproducts and quotient objects [= extremal epimorphisms] in $\mathcal{C}$.

(3) $\mathcal{A}$ has a monocoreflective hull, i.e. a smallest monocoreflective subcategory (of $\mathcal{C}$) containing $\mathcal{A}$, whose object class consists of all $\mathcal{C}$ -objects which are quotient objects of coproducts of $\mathcal{A}$ -objects (quotient objects and coproducts are formed in $\mathcal{C}$!).

1.9 Definitions. [1] A concrete category $(\mathcal{A},\mathcal{F})$ over $\mathcal{X}$ is called *initially complete* provided that $\mathcal{F}:\mathcal{A}\to\mathcal{X}$ is topological.

[2] If $(\mathcal{A},\mathcal{F})$ and $(\mathcal{B},\mathcal{G})$ are concrete categories over $\mathcal{X}$, then a functor $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ is called

a) *concrete* provided that $\mathcal{G}\circ\mathcal{H}=\mathcal{F}$,

b) *initiality preserving* provided that it is concrete and for each $\mathcal{F}$ -initial source $\big(A,(f_i:A\to A_i)_{i\in I}\big)$ in $\mathcal{A}$, the source $\big(\mathcal{H}(A),(\mathcal{H}(f_i):\mathcal{H}(A)\to\mathcal{H}(A_i))_{i\in I}\big)$ is $\mathcal{G}$ -initial in $\mathcal{B}$,

c) *initially dense* provided that it is concrete and for each $B\in|\mathcal{B}|$, there exists a $\mathcal{G}$ -initial source $\big(B,(g_i:B\to\mathcal{H}(A_i))_{i\in I}\big)$,

d) *finally dense* provided that it is concrete and for each $B\in|\mathcal{B}|$, there exists a $\mathcal{G}$ -final sink $\big((g_i:\mathcal{H}(A_i)\to B)_{i\in I},B\big)$.

[3] An *initial completion* of a concrete category $(\mathcal{A},\mathcal{F})$ is an initiality preserving initially dense full (concrete) embedding $\mathcal{H}:(\mathcal{A},\mathcal{F})\to(\mathcal{B},\mathcal{G})$ from $(\mathcal{A},\mathcal{F})$ into some initially complete category $(\mathcal{B},\mathcal{G})$. Occasionally $(\mathcal{B},\mathcal{G})$ is already called an initial completion of $(\mathcal{A},\mathcal{F})$ provided that $\mathcal{H}:(\mathcal{A},\mathcal{F})\to(\mathcal{B},\mathcal{G})$ is an initial completion of $(\mathcal{A},\mathcal{F})$.

[4] A finally-dense initial completion of a concrete category $(\mathcal{A},\mathcal{F})$, if it exists, is called a MACNEILLE *completion* of $(\mathcal{A},\mathcal{F})$.

1.10 Proposition (cf. [9; 6.1.3]). Let $\mathcal{H}:(\mathcal{A},\mathcal{F})\to(\mathcal{B},\mathcal{G})$ be a concrete functor which is full and finally dense. Then $\mathcal{H}$ is initiality preserving.

1.11 Example. Each poset $(S, \leq)$ defines a (small) category $\mathcal{A}$ by

1) $|\mathcal{A}| = S$ and

2) $[s, s']_{\mathcal{A}} = \begin{cases} \{(s,s')\} & \text{if } s \leq s' \\ \phi & \text{otherwise} . \end{cases}$

Then $\mathcal{A}$ is a concrete category over the category $\mathcal{X}$ consisting of exactly one object X and one morphism 1_X (the underlying functor $\mathcal{F} : \mathcal{A} \to \mathcal{X}$ is defined by $\mathcal{F}(s) = X$ for each $s \in |\mathcal{A}| = S$ and $\mathcal{F}(f) = 1_X$ for each $f \in \operatorname{Mor} \mathcal{A}$). If $N(S)$ denotes the set of all cuts in $(S, \leq)$ and $\subset$ the the set-theoretic inclusion, then there is an embedding $\mu_S : (S, \leq) \to (N(S), \subset)$ of $(S, \leq)$ into the complete lattice $(N(S), \subset)$, known as the classical MacNeille completion of $(S, \leq)$. Analogous to the construction of $(\mathcal{A}, \mathcal{F})$, a concrete category $(\mathcal{B}, \mathcal{G})$ over $\mathcal{X}$ is constructed from $(N(S), \subset)$. $\mathcal{H} : (\mathcal{A}, \mathcal{F}) \to (\mathcal{B}, \mathcal{G})$ defined by $\mathcal{H}(s) = \mu_S(s)$ for each $s \in S = |\mathcal{A}|$ and $\mathcal{H}((s,s')) = (\mu_S(s), \mu_S(s'))$ if $s \leq s'$ is a finally dense initial completion of $(\mathcal{A}, \mathcal{F})$, i.e. a MacNeille completion in the sense defined above.

§2 Relative (connectedness and) disconnectedness

2.1. For a topological category $\mathcal{C}$ the *category of pairs with respect to* $\mathcal{C}$ is denoted by $\mathcal{C}_{(2)}$, i.e.

(1) objects of $\mathcal{C}_{(2)}$ are pairs $((X,\xi),(Y,\eta))$ where (X,ξ) is an object in $\mathcal{C}$, Y a subset of X and η the initial $\mathcal{C}$ -structure with respect to $(Y, i, (X,\xi))$ where $i : Y \to X$ is the inclusion map.

(2) morphisms $f : ((X,\xi),(Y,\eta)) \to ((X',\xi'),(Y',\eta'))$ are morphisms $f : (X,\xi) \to (X',\xi')$ in $\mathcal{C}$ such that $f[Y] \subset Y'$.

For simplicity one often writes $(X, Y) \in |\mathcal{C}_{(2)}|$ instead of $((X,\xi),(Y,\eta)) \in |\mathcal{C}_{(2)}|$.
If $(X,Y) \in |\mathcal{C}_{(2)}|$ and $f : X \to X'$ is a $\mathcal{C}$ -morphism , then one writes usually $f \mid Y$ instead of $f \circ i$, where $i : Y \to X$ is the inclusion map.

2.2 Definitions. [1] Let $\mathcal{P}$ be a subclass of $|\mathcal{C}|$:

(a) Let $(X,Y) \in |\mathcal{C}_{(2)}|$. Y is called $\mathcal{P}$ *-connected* with respect to X iff $f \mid Y$ is constant for each $P \in \mathcal{P}$ and each $\mathcal{C}$ -morphism $f : X \to P$.

(b) $C_{rel}\mathcal{P} = \{(X,Y) \in |\mathcal{C}_2| : Y \text{ is } \mathcal{P}\text{-connected with respect to } X\}$.

(c) $\mathcal{K} \subset |\mathcal{C}_{(2)}|$ is called a *relative connectedness* iff $\mathcal{K} = C_{rel}\mathcal{P}$ for some $\mathcal{P} \subset |\mathcal{C}|$.

[2] Let $\mathcal{K}$ be a subclass of $|\mathcal{C}_{(2)}|$:

(a) $D_{rel}\mathcal{K} = \{Z \in |\mathcal{C}| : f \mid Y$ is constant for each $\mathcal{C}$ -morphism $f : X \to Z$ and each $Y \subset X$ satisfying $(X,Y) \in \mathcal{K}\}$.

(b) $\mathcal{P} \subset |\mathcal{C}|$ is called a *relative disconnectedness* iff $\mathcal{P} = D_{rel}\mathcal{K}$ for some $\mathcal{K} \subset |\mathcal{C}_{(2)}|$.

2.3 Remarks. [1] $Q = D_{rel} C_{rel}$ is a hull operator, i.e. Q is extensive ($\mathcal{P} \subset Q\mathcal{P}$ for each $\mathcal{P} \subset | \mathcal{C} |$), isotonic ($\mathcal{P} \subset \mathcal{Q} \subset | \mathcal{C} |$ implies $Q\mathcal{P} \subset Q\mathcal{Q}$) and idempotent ($QQ = Q$). [Analogously, $P = C_{rel} D_{rel}$ is a hull operator.]

[2] A subclass $\mathcal{P}$ of $| \mathcal{C} |$ is a relative disconnectedness iff $\mathcal{P} = Q\mathcal{P}$ (Analogously, $\mathcal{K} \subset | \mathcal{C}_{(2)} |$ is a relative connectedness iff $\mathcal{K} = P\mathcal{K}$).

[3] There is a one-to-one correspondence between the relative connectednesses of $| \mathcal{C}_{(2)} |$ and the relative disconnectednesses of $| \mathcal{C} |$ which converts the inclusion relation (GALOIS *correspondence)*, and is obtained by the operators C_{rel} and D_{rel} .

[4] Let $\mathcal{C}$ be a topological category and $\mathcal{P} \subset | \mathcal{C} |$. For each $X \in | \mathcal{C} |$ and each $x \in X$ the union K_x of all Y with $(X, Y) \in C_{rel}\mathcal{P}$ and $x \in Y$ is $\mathcal{P}$ -connected with respect to X and called the $\mathcal{P}$ *-quasicomponent* of X containing x . Obviously, the $\mathcal{P}$ -quasicomponents of X form a decomposition of X . In particular, if $\mathcal{C} = \mathbf{Top}$ and $\mathcal{P} = \{D_2\}$ (D_2 : two-point discrete topological space), then the $\mathcal{P}$ -quasicomponents of a topological space X are nothing else than the quasicomponents of X in the usual sense, originally defined by HAUSDORFF [3].

2.4 Proposition. Let $\mathcal{C}$ be a topological category, $\mathcal{P} \subset | \mathcal{C} |$ and $X \in | \mathcal{C} |$. Then the following are equivalent:

(1) $X \in Q\mathcal{P}$.

(2) For each $x \in X$,the $\mathcal{P}$ -quasicomponent K_x of X containing x is a singleton.

(3) For any two distinct elements $x, y \in X$ there exists an object $P \in \mathcal{P}$ and a $\mathcal{C}$ -morphism $f : X \to P$ such that $f(x) \neq f(y)$.

Proof. The equivalence of (2) and (3) is obvious.

(1) $\Rightarrow$ (2). The identity $1_X : X \to X$ is a $\mathcal{C}$ -morphism, and since $(X, K_x) \in C_{rel}\mathcal{P}$ for each $x \in X$, $1_X \mid K_x : K_x \to X$ is constant, because $X \in D_{rel} C_{rel} \mathcal{P}$. Thus, K_x is a singleton.

(2) $\Rightarrow$ (1). Let $f : Y \to X$ be a $\mathcal{C}$ -morphism and $Z \subset Y$ such that $(Y, Z) \in C_{rel}\mathcal{P}$. Then, obviously, $(X, f[Z]) \in C_{rel}\mathcal{P}$. Thus, $f[Z]$ is contained in a $\mathcal{P}$ -quasicomponent of X which is a singleton. Therefore $f \mid Z$ is constant.

2.5 Definition. The elements of $Q\mathcal{P}$ are called *totally* $\mathcal{P}$ *-separated.*

2.6 Examples for the category Top.

[1] a) Let $\mathcal{P} = \{S\}$ where S is the SIERPINSKI space $\left(\{0,1\}, \left\{ \emptyset, \{0\}, \{0,1\}\right\}\right)$. Then $Q\{S\} = \{T_0\text{-spaces}\}$.

b) Let $\mathcal{P} = \{T_0\text{-spaces}\} : Q\mathcal{P} = \{T_0\text{-spaces}\}$ (Apply a) and note $QQ = Q$!).

[2] a) $\mathcal{P} = \{\text{spaces with the cofinite topology}\} : Q\mathcal{P} = \{T_1\text{-spaces}\}$.

b) $\mathcal{P} = \{T_1\text{-spaces}\} : Q\mathcal{P} = \{T_1\text{-spaces}\}$ (Apply a) and note $QQ = Q$!).

[3] $\mathcal{P} = \{T_2\text{-spaces}\} : Q\mathcal{P} = \{T_2\text{-spaces}\}$.

[4] $\mathcal{P} = \{$URYSOHN -spaces$\} : Q\mathcal{P} = \{$URYSOHN -spaces$\}$.

[5] $\mathcal{P} = \{\mathbb{R}\}$, where $\mathbb{R}$ denotes the set of real numbers endowed with the usual topology or $\mathcal{P} = \{[0,1]\}$ ($[0,1]$: closed unit interval endowed with the topology induced by $\mathbb{R}$) : $Q\mathcal{P} = \{$completely HAUSDORFF SPACES$\}$.

[6] $\mathcal{P} = \{D_2\} : Q\mathcal{P} = \{$totally separated spaces$\}$.

2.7 Theorem. Let $\mathcal{C}$ be a topological category, $\mathcal{P}$ a subclass of $|\mathcal{C}|$ such that the full subcategory $\mathcal{A}$ of $\mathcal{C}$ defined by $|\mathcal{A}| = \mathcal{P}$ is isomorphism-closed. Then the following are equivalent:

(1) $\mathcal{P}$ is a relative disconnectedness.

(2) $\mathcal{A}$ is an extremal epireflective subcategory of $\mathcal{C}$.

Proof. (1) $\Rightarrow$ (2). Since $\mathcal{P} = Q\mathcal{P}$ it suffices to prove that $Q\mathcal{P}$ is closed under formation of weak subobjects and products in $\mathcal{C}$: Let $(f_i : X \to X_i)_{i\in I}$ be a mono-source in $\mathcal{C}$ such that $X_i \in Q\mathcal{P}$ for each $i \in I$. If $a, b \in X$ with $a \neq b$, then there is some $i \in I$ such that $f_i(a) \neq f_i(b)$. Since $X_i \in Q\mathcal{P}$, there exists some $P \in \mathcal{P}$ and some $\mathcal{C}$ -morphism $f : X_i \to P$ with $f(f_i(a)) \neq f(f_i(b))$. Thus $h = f \circ f_i : X \to P$ is a $\mathcal{C}$ -morphism such that $h(a) \neq h(b)$. i.e. $X \in Q\mathcal{P}$ (cf. 2.4.).

(2) $\Rightarrow$ (1). It suffices to prove $Q\mathcal{P} \subset \mathcal{P}$ ($\mathcal{P} \subset Q\mathcal{P}$ is always true). If $X \in Q\mathcal{P}$, then there exists an $\mathcal{A}$ -reflection $e : X \to Y$ which is an extremal epimorphism. Let $x, y \in X$ such that $x \neq y$. Then there exists some $P \in \mathcal{P}$ and some $\mathcal{C}$ -morphism $f : X \to P$ with $f(x) \neq f(y)$. Since e is an $\mathcal{A}$ -reflection there exists a unique $\mathcal{C}$ -morphism $\overline{f} : Y \to P$ with $\overline{f} \circ e = f$. Hence $e(x) \neq e(y)$. Consequently, e is injective, i.e. a monomorphism. Thus e is an isomorphism. Since $\mathcal{A}$ is isomorphism-closed, $X \in |\mathcal{A}| = \mathcal{P}$.

§3 Relations between topological categories, monotopological categories and MACNEILLE completions

3.1 Theorem (HERRLICH, SCHWARZ, ALDERTON). For a concrete category $\mathcal{C}$ over Set the following are equivalent:

(1) $\mathcal{C}$ is monotopological.

(2) $\mathcal{C}$ is (concretely isomorphic to) an epireflective subcategory of a topological category.

(3) $\mathcal{C}$ is (concretely isomorphic to) an epireflective subcategory of a topological category which is a MACNEILLE completion of $\mathcal{C}$.

(4) $\mathcal{C}$ is (concretely isomorphic to) an extremal epireflective subcategory of a topological category.

(5) $\mathcal{C}$ is (concretely isomorphic to) an extremal epireflective subcategory of a topological category which is a MACNEILLE completion of $\mathcal{C}$.

Proof. It suffices to prove the implications "(2) $\Rightarrow$ (1)" and "(1) $\Rightarrow$ (5)":

(2) $\Rightarrow$ (1). Since $\mathcal{C}$ is an epireflective subcategory of a topological category $\mathcal{B}$, it coincides with its epireflective hull whose object class consists of all $X \in |\mathcal{B}|$ for which there exists an initial mono-source $(f_i : X \to X_i)_{i\in I}$ with $X_i \in |\mathcal{C}|$. If X is a set, $((X_i, \xi_i))_{i\in I}$ a family of $\mathcal{C}$-objects and $(f_i : X \to X_i)_{i\in I}$ a mono-source in **Set**, then there exists a unique initial $\mathcal{B}$-structure ξ on X with respect to $(X, f_i, (X_i, \xi_i), I)$. Thus $(f_i : (X,\xi) \to (X_i, \xi_i))_{i\in I}$ is an initial mono-source in $\mathcal{B}$ and (X,ξ) belongs to $|\mathcal{C}|$. Therefore, ξ is the desired initial $\mathcal{C}$-structure. The remaining statements follow immediately from the corresponding properties of $\mathcal{B}$.

(1) $\Rightarrow$ (5). Let $\mathcal{B}$ be the following category: objects are all triples (X, e, A) such that $X \in |\mathbf{Set}|$, $A \in |\mathcal{C}|$ and $e : X \to \mathcal{F}_\mathcal{C}(A)$ is an epimorphism where $\mathcal{F}_\mathcal{C} : \mathcal{C} \to \mathbf{Set}$ denotes the underlying functor; morphisms from (X, e, A) to (X', e', A') are all pairs (f, g) with $f : X \to X'$, $g : A \to A'$ and $e' \circ f = \mathcal{F}_\mathcal{C}(g) \circ e$; and the composition of morphisms is defined componentwise. A functor $\mathcal{F}_\mathcal{B} : \mathcal{B} \to \mathbf{Set}$ is defined by $\mathcal{F}_\mathcal{B}((X, e, A)) = X$ and $\mathcal{F}_\mathcal{B}((f, g)) = f$. Then $\mathcal{F}_\mathcal{B}$ is topological (cf. [4; 9.1.]). The full subcategory $\mathcal{B}'$ of $\mathcal{B}$ defined by

$$|\mathcal{B}'| = \{(X, e, A) \in |\mathcal{B}| : \mathcal{F}_\mathcal{C}(A) \subset X\}$$

obviously satisfies all properties of a topological category with the exception of the uniqueness of initial structures. Now an equivalence relation $\sim$ on $|\mathcal{B}'|$ is defined by

$$(X, e, A) \sim (X', e', A') \quad \text{iff} \quad \begin{cases} X = X' & \text{and there exists an isomorphism} \\ g : A \to A' & \text{with } \mathcal{F}_\mathcal{C}(g) \circ e = e' . \end{cases}$$

If $|\mathcal{B}''|$ is a system of representatives of $\sim$ containing $|\mathcal{C}''| = \left\{(\mathcal{F}_\mathcal{C}(A), 1_{\mathcal{F}_\mathcal{C}(A)}, A) : A \in |\mathcal{C}|\right\}$ (note, that $\mathcal{F}_\mathcal{C}$ is amnestic), then the corresponding full subcategory $\mathcal{B}''$ of $\mathcal{B}'$ is a topological category. Moreover, the full subcategory $\mathcal{C}''$ of $\mathcal{B}'$ defined by $|\mathcal{C}''|$ as above is an isomorphism-closed extremal epireflective subcategory of $\mathcal{B}''$, where (obviously) $(e, 1_A) : (X, e, A) \to \left(\mathcal{F}_\mathcal{C}(A), 1_{\mathcal{F}_\mathcal{C}(A)}, A\right)$ is the extremal epireflection of $(X, e, A) \in |\mathcal{B}''|$ with respect to $\mathcal{C}''$. A (concrete) functor $\mathcal{H} : \mathcal{C} \to \mathcal{B}''$ is defined by $\mathcal{H}(A) = \left(\mathcal{F}_\mathcal{C}(A), 1_{\mathcal{F}_\mathcal{C}(A)}, A\right)$ for each $A \in |\mathcal{C}|$ and $\mathcal{H}(f) = (\mathcal{F}_\mathcal{C}(f), f)$ for each $f \in \operatorname{Mor} \mathcal{C}$. It is easy to check that $\mathcal{H}$ is a full embedding. Then $\mathcal{H}' : \mathcal{C} \to \mathcal{C}''$ defined by $\mathcal{H}'(A) = \mathcal{H}(A)$ for each $A \in |\mathcal{C}|$ and $\mathcal{H}'(f) = \mathcal{H}(f)$ for each $f \in \operatorname{Mor} \mathcal{C}$ is an isomorphism. Let $\mathcal{D}$ be the monocoreflective (= bicoreflective) hull of $\mathcal{C}''$ in $\mathcal{B}''$ (cf. 1.8 (3)). Then, obviously, the inclusion functor $\mathcal{I} : \mathcal{C}'' \to \mathcal{D}$ is finally dense and $\mathcal{D}$ is topological (cf. 1.6). Moreover, $\mathcal{I} : \mathcal{C}'' \to \mathcal{D}$ is initially dense and $\mathcal{C}''$ is extremal epireflective in $\mathcal{D}$ (note: $(e, 1_A) : (X, e, A) \to (\mathcal{F}_\mathcal{C}(A), 1_{\mathcal{F}_\mathcal{C}(A)}, A)$) is the extremal epireflection of $(X, e, A) \in |\mathcal{D}|$ with respect to $\mathcal{C}''$ and (e, A) is the initial $\mathcal{D}$-structure on X with respect to $(e, 1_A)$). By 1.10, $\mathcal{I} : \mathcal{C}'' \to \mathcal{D}$ is initiality preserving. Thus $\mathcal{I} : \mathcal{C}'' \to \mathcal{D}$ is a MACNEILLE completion of $\mathcal{C}''$.

3.2 Corollary. A concrete category $\mathcal{C}$ over **Set** is monotopological iff its object class $|\mathcal{C}|$ is a relative disconnectedness with respect to a topological category which is a MACNEILLE completion of $\mathcal{C}$.

Proof. Apply 2.7 and 3.1.

3.3 Example. Let $\mathcal{D}$ be the category of regular spaces (resp. completely regular spaces) [and continuous maps] and $\mathcal{C}$ the category of T_3-spaces [= regular T_1-spaces] (resp. TYCHONOFF spaces [= completely regular T_1 -spaces]). Then $\mathcal{D}$ is topological and a MACNEILLE completion of $\mathcal{C}$. Furthermore, $|\mathcal{C}|$ is a relative disconnectedness (i.e. $|\mathcal{C}| = Q\ |\mathcal{C}|$) with respect to $\mathcal{D}$. (Thus, $\mathcal{C}$ is monotopological).

3.4 Remarks. [1] ALDERTON [2; 2.2] has shown that the MACNEILLE completion $\widehat{\mathcal{H}} = \mathcal{I} \circ \mathcal{H}' : \mathcal{C} \to \mathcal{D}$ of a monotopological category $\mathcal{C}$ as constructed above has the following additional property: If $\mathcal{K} : \mathcal{C} \to \widehat{\mathcal{C}}$ is any initial completion of $\mathcal{C}$, then there exists a full concrete embedding $\mathcal{E} : \widehat{\mathcal{C}} \to \mathcal{D}$ such that $\mathcal{E} \circ \mathcal{K} = \widehat{\mathcal{H}}$. Hence, any initial completion of a monotopological category $\mathcal{C}$ is a MACNEILLE completion of $\mathcal{C}$.

[2] An object X of a topological category $\mathcal{C}$ is called a T_0 *-object* provided that each $\mathcal{C}$ -morphism from a two-point indiscrete object (i.e. a two-point set endowed with the initial $\mathcal{C}$ -structure with respect to the empty index class) to X is constant. Obviously, the T_0 -objects of $\mathcal{C}$ form a relative disconnectedness. WECK-SCHWARZ [11] has shown, that the object class of a properly monotopological category $\mathcal{C}$ (i.e. a monotopological category which is not topological) consists of all T_0 -objects of its MACNEILLE completion being a topological category.

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