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Strong Integrality of Inversion Subgroups of Kac-Moody Groups

Let A be a symmetrizable generalized Cartan matrix with corresponding Kac-Moody algebra $\mathfrak{g}$ over $\mathbb{Q}$. Let $V = V^\lambda$ be an integrable highest weight $\mathfrak{g}$-module with dominant regular integral weight λ and representation $\rho : \mathfrak{g} \to \mathrm{End}\,(V)$, and let $V_{\mathbb{Z}} = V^\lambda_{\mathbb{Z}}$ be a $\mathbb{Z}$-form of V. Let $G_V(\mathbb{Q})$ be the associated minimal Kac-Moody group generated by the automorphisms $\exp(t\rho(e_i))$ and $\exp(t\rho(f_i))$ of V, where e_i and f_i are the Chevalley-Serre generators and $t \in \mathbb{Q}$. Let $G(\mathbb{Z})$ be the group generated by $\exp(t\rho(e_i))$ and $\exp(t\rho(f_i))$ for $t \in \mathbb{Z}$. Let $\Gamma(\mathbb{Z})$ be the Chevalley subgroup of $G_V(\mathbb{Q})$, that is, the subgroup that stabilizes the lattice $V_{\mathbb{Z}}$ in V. For a subgroup M of $G_V(\mathbb{Q})$, we say that M is integral if $M \cap G(\mathbb{Z}) = M \cap \Gamma(\mathbb{Z})$ and that M is strongly integral if there exists $v \in V_{\mathbb{Z}}$ such that $g \cdot v \in V_{\mathbb{Z}}$ implies $g \in G(\mathbb{Z})$ for all $g \in M$. We prove strong integrality of inversion subgroups $U_{(w)}$ of $G_V(\mathbb{Q})$ for w in the Weyl group, where $U_{(w)}$ is the group generated by positive real root groups that are flipped to negative root groups by w^{-1}. We use this to prove strong integrality of subgroups of the unipotent subgroup U of $G_V(\mathbb{Q})$ that are generated by commuting real root groups. When A has rank 2, this gives strong integrality of subgroups U_1 and U_2 where $U = U_1 * U_2$ and each U_i is generated by 'half' the positive real roots.

Keywords: Kac-Moody groups, Chevalley groups, integrality.

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