On the Areas of Level Sets in Compact Connected Sublattices of Three-Dimensional Euclidean Space

As is well-known the three-dimensional Euclidean space \( \mathbb{R}^3 \), equipped with the order relation \((x_1, x_2, x_3) \leq (x'_1, x'_2, x'_3)\) if \( x_i \leq x'_i \) for \( i = 1, 2, 3 \), is a distributive, topological lattice. Let \( L \) be a compact, connected sublattice of \( \mathbb{R}^3 \). For \((x_1, x_2, x_3) \in L\) we define \( \lambda(x_1, x_2, x_3) = x_1 + x_2 + x_3 \) and for \( r \in \mathbb{R} \) we let \( L_r = \{(x_1, x_2, x_3) \in L : \lambda(x_1, x_2, x_3) = r\} \). If \( \mu_L(r) \) denotes the surface area of \( L_r \), then we show that the function \( r \mapsto \mu_L(r) \) is continuously differentiable, and that the value of \( \mu'_L(r) \) can be computed in two different ways: Either as an integral of a certain function over the boundary of \( L_r \), or as the value of the expression \( \sqrt{3} (\lambda(\sup L_r) + \lambda(\inf L_r) - 2r) \).

**Keywords:** Level sets and rank functions, sublattices of \( \mathbb{R}^3 \), integral formulas.

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