On the Characterization of Trace Class Representations and Schwartz Operators

We collect several characterizations of unitary representations \((\pi, \mathcal{H})\) of a finite dimensional Lie group \(G\) which are trace class, i.e., for each compactly supported smooth function \(f\) on \(G\), the operator \(\pi(f)\) is trace class. In particular we derive the new result that, for some \(m \in \mathbb{N}\), all operators \(\pi(f), f \in C^m_c(G)\), are trace class. As a consequence the corresponding distribution character \(\theta_\pi\) is of finite order. We further show \(\pi\) is trace class if and only if every operator \(A\), which is smoothing in the sense that \(A\mathcal{H} \subseteq \mathcal{H}^\infty\), is trace class and that this in turn is equivalent to the Fréchet space \(\mathcal{H}^\infty\) being nuclear, which in turn is equivalent to the realizability of the Gaussian measure of \(\mathcal{H}\) on the space \(\mathcal{H}^{-\infty}\) of distribution vectors. Finally we show that, even for infinite dimensional Fréchet-Lie groups, \(A\) and \(A^*\) are smoothing if and only if \(A\) is a Schwartz operator, i.e., all products of \(A\) with operators from the derived representation are bounded.

Keywords: Trace class representation, smoothing operator, Schwartz operator, unitary representation.

MSC: 22E45, 22E66