Spherical Functions: The Spheres versus the Projective Spaces

We establish a close relationship between the spherical functions of the $n$-dimensional sphere $S^n \cong \text{SO}(n+1)/\text{SO}(n)$ and those of the $n$-dimensional real projective space $P^n(\mathbb{R}) \cong \text{SO}(n+1)/\text{O}(n)$. In fact, for $n$ odd a function on $\text{SO}(n+1)$ is an irreducible spherical function of some type $\pi \in \hat{\text{SO}}(n)$ if and only if it is an irreducible spherical function of some type $\gamma \in \hat{\text{O}}(n)$. When $n$ is even this is also true for certain types, and in the other cases we exhibit a clear correspondence between the irreducible spherical functions of both pairs $(\text{SO}(n+1), \text{SO}(n))$ and $(\text{SO}(n+1), \text{O}(n))$. Summarizing, to find all spherical functions of one pair is equivalent to do so for the other pair. 

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