The Structure of \( H \)-(co)module Lie algebras

Let \( L \) be a finite dimensional Lie algebra over a field of characteristic 0. Then by the original Levi theorem, \( L = B \oplus R \) where \( R \) is the solvable radical and \( B \) is some maximal semisimple subalgebra. We prove that if \( L \) is an \( H \)-(co)module algebra for a finite dimensional (co)semisimple Hopf algebra \( H \), then \( R \) is \( H \)-(co)invariant and \( B \) can be chosen to be \( H \)-(co)invariant too. Moreover, the nilpotent radical \( N \) of \( L \) is \( H \)-(co)invariant and there exists an \( H \)-sub(co)module \( S \subseteq R \) such that \( R = S \oplus N \) and \( [B,S] = 0 \). In addition, the \( H \)-(co)invariant analog of the Weyl theorem is proved. In fact, under certain conditions, these results hold for an \( H \)-comodule Lie algebra \( L \), even if \( H \) is infinite dimensional.

In particular, if \( L \) is a Lie algebra graded by an arbitrary group \( G \), then \( B \) can be chosen to be graded, and if \( L \) is a Lie algebra with a rational action of a reductive affine algebraic group \( G \) by automorphisms, then \( B \) can be chosen to be \( G \)-invariant. Also we prove that every finite dimensional semisimple \( H \)-(co)module Lie algebra over a field of characteristic 0 is a direct sum of its minimal \( H \)-(co)invariant ideals.

**Keywords**: Lie algebra, stability, Levi decomposition, radical, grading, Hopf algebra, Hopf algebra action, \( H \)-module algebra, \( H \)-comodule algebra.

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