Picard Groups of Siegel Modular 3-Folds and \( \theta \)-Liftings

We show that the Humbert surfaces rationally generate the Picard groups of Siegel modular threefolds. This involves three ingredients: (1) R. Weissauer’s determination of these Picard groups in terms of theta lifting from cusp forms of weight \( 5/2 \) on \( \widetilde{SL}_2(\mathbb{R}) \) to automorphic forms on \( \text{Sp}_4(\mathbb{R}) \). (2) The theory of special cycles due to Kudla/Millson and Tong/Wang relating cohomology defined by automorphic forms to that defined by certain geometric cycles. (3) Results of R. Howe about the structure of the oscillator representation in this situation.

Keywords: Siegel modular threefold, Picard group, theta lifting.

MSC: 14G35; 11F46, 11F27, 14C22, 11F23