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The Minimal Representation of the Conformal Group and Classical Solutions to the Wave Equation

Using an idea of Dirac, we give a geometric construction of a unitary lowest weight representation $\mathcal{H}^+$ and a unitary highest weight representation $\mathcal{H}^-$ of a double cover of the conformal group $\mathrm{SO}(2, n+1)_0$ for every $n \geq 2$. The smooth vectors in $\mathcal{H}^+$ and $\mathcal{H}^-$ consist of complex-valued solutions to the wave equation $\Box f = 0$ on Minkowski space $\mathbb{R}^{1,n} = \mathbb{R} \times \mathbb{R}^n$ and the invariant product is the usual Klein-Gordon product. We then give explicit orthonormal bases for the spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ consisting of weight vectors; when n is odd, our bases consist of rational functions. Furthermore, we show that if $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^{1,n})$ are real-valued Schwartz functions and $u \in \mathcal{C}^\infty(\mathbb{R}^{1,n})$ is the (real-valued) solution to the Cauchy problem $\Box u = 0$, $u(0,x) = \Phi(x)$, $\partial_t u(0,x) = \Psi(x)$, then there exists a unique real-valued $v \in \mathcal{C}^\infty(\mathbb{R}^{1,n})$ such that $u + iv \in \mathcal{H}^+$ and $u - iv \in \mathcal{H}^-$.

Keywords: Conformal group, minimal representation, wave equation, classical solutions, Cauchy problem.

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