Geometric Structures on Lie Groups with Flat Bi-Invariant Metric

Let $L \subset V = \mathbb{R}^{k,l}$ be a maximally isotropic subspace. It is shown that any simply connected Lie group with a bi-invariant flat pseudo-Riemannian metric of signature $(k,l)$ is 2-step nilpotent and is defined by an element $\eta \in \Lambda^3L \subset \Lambda^3V$.

If $\eta$ is of type $(3,0) + (0,3)$ with respect to a skew-symmetric endomorphism $J$ with $J^2 = \varepsilon \text{Id}$, then the Lie group $\mathcal{L}(\eta)$ is endowed with a left-invariant nearly Kähler structure if $\varepsilon = -1$ and with a left-invariant nearly para-Kähler structure if $\varepsilon = +1$. This construction exhausts all complete simply connected flat nearly (para-)Kähler manifolds. If $\eta \neq 0$ has rational coefficients with respect to some basis, then $\mathcal{L}(\eta)$ admits a lattice $\Gamma$, and the quotient $\Gamma \backslash \mathcal{L}(\eta)$ is a compact inhomogeneous nearly (para-)Kähler manifold. The first non-trivial example occurs in six dimensions.

Keywords: Flat Lie-groups, bi-invariant metrics, nearly para-Kähler manifolds, flat almost para-Hermitian manifolds, almost para-complex structures.

MSC: 53C50, 53C15