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Integral Inequalities for Infimal Convolution and Hamilton-Jacobi Equations

Let $f, g : \mathbb{R}^N \to (-\infty, \infty]$ be Borel measurable, bounded below and such that $\inf f + \inf g \geq 0$. We prove that with $m_{f,g} := (\inf f - \inf g)/2$, the inequality

$$||(f - m_{f,g})^{-1}||_\phi + ||(g + m_{f,g})^{-1}||_\phi \leq 4||(f \Box g)^{-1}||_\phi$$

holds in every Orlicz space L_ϕ, where $f \Box g$ denotes the infimal convolution of f and g and where $||\cdot||_\phi$ is the Luxemburg norm (i.e., the L^p norm when $L_\phi = L^p$). Although no genuine reverse inequality can hold in any generality, we also prove that such reverse inequalities do exist in the form

$$||(f \Box g)^{-1}||_\phi \leq 2^{N-1}(||(\check{f} - m_{f,g})^{-1}||_\phi + ||(\check{g} + m_{f,g})^{-1}||_\phi),$$

where $\check{f}$ and $\check{g}$ are suitable transforms of f and g introduced in the paper and reminiscent of, yet very different from, nondecreasing rearrangement.
Similar inequalities are proved for other extremal operations and applications are given to the long-time behavior of the solutions of the Hamilton-Jacobi and related equations.

Keywords: Brunn-Minkowski inequality, enclosing ball, Hamilton-Jacobi equations, infimal convolution, Orlicz space, rearrangement.

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