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On the Local Minimizers of the Mahler Volume

We focus on the analysis of local minimizers of the Mahler volume, that is to say the local solutions to the problem

$$\min\{M(K) := |K||K^\circ| \;/\; K \subset \mathbb{R}^d \text{ open and convex, } K = -K\},$$

where $K^\circ := \{\xi \in \mathbb{R}^d; \forall x \in K, x \cdot \xi < 1\}$ is the polar body of K, and $|\cdot|$ denotes the volume in $\mathbb{R}^d$. According to a famous conjecture of Mahler the cube is expected to be a global minimizer for this problem.
In this paper we express the Mahler volume in terms of the support functional of the convex body, which allows us to compute first and second derivatives of the obtained functional. We deduce from these computations a concavity property of the Mahler volume which seems to be new. As a consequence of this property, we retrieve a result which supports the conjecture, namely that any local minimizer has a Gauss curvature that vanishes at any point where it is defined (first proven by S. Reisner, C. Schütt and E. M. Werner [Mahler's conjecture and curvature, Int. Math. Res. Not. IMRN 2012, no. 1, 1-16]). Going more deeply into the analysis in the two-dimensional case, we generalize the concavity property of the Mahler volume and also deduce a new proof that any local minimizer must be a parallelogram (proven by K. J. Böröczky, E. Makai, M. Meyer and S. Reisner [On the volume product of planar polar convex bodies – Lower estimates with stability, Studia Scientiarum Mathematicarum Hungarica 50 (2013) 159–198]).

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