Journal of Convex Analysis 22 (2015) 769–796

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Monge-Ampère Type Function Splittings

Given convex $u \in C(\bar{\Omega})$ with Monge-Ampère measure Mu, and finite Borel measures μ and ν satisfying $\mu + \nu = Mu$, consider the problem of determing a 'splitting' $u = v + w$ for u where $v, w \in C(\bar{\Omega})$ are convex functions satisfying $Mv = \mu$, $Mw = \nu$, so that $Mu = M(v + w) = Mv + Mw$. It is shown that although this problem is not in general solvable, a best L^p approximation $v^* + w^*$ for u may always be found. In particular, letting $U = \sup_{(v,w)\in\mathcal{F}} (v + w)$, there exist optimal sums $v^* + w^*$ achieving $\inf_{(v,w)\in\mathcal{F}} \|u - (v+w)\|_p$ and $\inf_{(v,w)\in\mathcal{F}} \|U - (v+w)\|_p$, $p \geq 1$, for appropriately constrained classes $\mathcal{F}$ of feasible pairs (v, w) of convex functions satisfying $Mv = \mu$, $Mw = \nu$ and $v + w = u$ on $\partial\Omega$. Moreover, U may be written as $U = \bar{v} + \bar{w}$ within $\bar{\Omega}$, $(\bar{v}, \bar{w}) \in \mathcal{F}$. The analysis depends upon basic properties of convex functions and the measures they determine.

We also consider the related problem of characterizing functions $u \in W^{2,n}(\Omega)$ which may be realized as differences $u = v - w$ of convex functions $v, w \in W^{2,n}(\Omega)$ with $Mu = Mv - Mw$. Here Mu is the signed measure defined by $dMu = \det\, D^2u\, dx$. Letting $U^- = \sup_{(v,w)\in\mathcal{F}}(v - w)$ and $U_- = \inf_{(v,w)\in\mathcal{F}}(v - w)$, we show that optimal differences $v^* - w^*$ exist for the problems $\inf_{(v,w)\in\mathcal{F}} \|u - (v - w)\|_p$, $\inf_{(v,w)\in\mathcal{F}} \|U^- - (v - w)\|_p$ and $\inf_{(v,w)\in\mathcal{F}} \|U_- - (v - w)\|_p$. Also, $U^- = v^- - w^-$ and $U_- = v_- - w_-$ for appropriate pairs $(v^-, w^-), (v_-, w_-) \in \mathcal{F}$.

Finally, the relaxed problem of finding $v + w = u$ for general Mv and Mw with $Mv + Mw = Mu$ (no fixed μ and ν), is considered. Topological properties of the collection of these relaxed splitting pairs (v, w), and those for the unrelaxed problem, for a given u, are developed.

Keywords: Monge-Ampere equations, additive solution, optimization characterizations, convex functions.

MSC: 20M05; 03D40