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Generalized Steffensen Inequalities and Their Optimal Constants

If $\Phi : [0, \infty) \to \mathbb{R}$ is convex and continuous with $\Phi(0) = 0$ and if $q \in (1, \infty)$, $q' := \frac{q}{q-1}$, we first prove that the inequality

$$\Phi\left(\int_0^\infty f(r)dr\right) \leq C \int_0^\infty f(r)\Phi'(r^{1/q'})dr$$

for every $f \in L^q(0, \infty)$, $f \geq 0$ with $||f||_q \leq 1$ holds when $C = 1$. In general, both sides may be $\pm\infty$. Related inequalities for $f \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, $f \neq 0$ are derived. This inequality is independent of Jensen's inequality and, when $q = \infty$, it is an elaboration on an inequality of Steffensen which was discussed elsewhere by the author.
The next goal of the paper is to identify the range of the admissible constants C and, in particular, to characterize the optimal constant when $\Phi \geq 0$ or $\Phi \leq 0$. It turns out that $C = 1$ is "almost always" optimal, at least in a restricted sense, but not always when $q < \infty$: Given q, the admissible constants lie on an interval containing 1 whose left (right) endpoint is the supremum (infimum) of a function defined on some (left/right dependent) subset of $\mathbb{R}^2$.
If $q = 2$, these extrema can be calculated in a number of examples. Among other things, this reveals that $C = 1$ need not be optimal when $\Phi \geq 0$ and $\Phi'_+(0) = 0$ or when $\Phi \leq 0$ and $\Phi'_+(0) = -\infty$.

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