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Universal Approximants of the Riemann Zeta-Function

CMFT 9 No.1 (2009), 145–159. [ISSN 1617-9447]

Abstract. The Riemann zeta-function $\zeta(z)$ has the following well-known properties, cf. the excellent survey of Steuding [10]:
(i) it is holomorphic in the complex plane except for a simple pole at $z = 1$ with residue 1;
(ii) the symmetry relation $\zeta(z) = \overline{\zeta(\bar{z})}$ holds for $z \neq 1$;
(iii) the functional equation $\zeta(z)\Gamma(z/2)\pi^{-z/2} = \zeta(1-z)\Gamma((1-z)/2)\pi^{-(1-z)/2}$ holds;
(iv) it has a universality property due to Voronin [11].
The aim of this paper is to show that arbitrarily close approximations of the Riemann zeta-function which satisfy (i)–(iv) may have a different universal property. Consequently, these approximations do not satisfy the Riemann hypothesis. This extends a result due to Gauthier and Zeron [6].

Furthermore, we show that the set of all "Birkhoff-universal" functions satisfying (i)–(iii) is a dense G_δ-set in the set of all functions satisfying (i)–(iii).

Keywords. Universality, tangential approximation, Riemann zeta-function.

2000 MSC. 11M06, 30E10.

Received. October 16, 2007, in revised form January 21, 2008.

Published online. April 30, 2008.

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