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Sharp Bohr Type Real Part Estimates

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Abstract. We consider analytic functions f in the unit disk $\mathbb{D}$ with Taylor coefficients $c_0, c_1, \dots$ and derive estimates with sharp constants for the l_q-norm (quasi-norm for $0 < q < 1$) of the remainder of their Taylor series, where $q \in (0, \infty]$. As the main result, we show that given a function f with $\operatorname{Re} f$ in the Hardy space $h_1(\mathbb{D})$ of harmonic functions on $\mathbb{D}$, the inequality

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \, \| \operatorname{Re} f \|_{h_1}$$

holds with the sharp constant, where $r = |z| < 1$, $m \ge 1$. This estimate implies sharp inequalities for l_q-norms of the Taylor series remainder for bounded analytic functions, analytic functions with bounded $\operatorname{Re} f$, analytic functions with $\operatorname{Re} f$ bounded from above, as well as for analytic functions with $\operatorname{Re} f > 0$. In particular, we prove that

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \operatorname{Re}(f(\zeta) - f(0)).$$

As corollary of the above estimate with $\| \operatorname{Re} f \|_{h_1}$ in the right-hand side, we obtain some sharp Bohr type modulus and real part inequalities.

Keywords. Taylor series, Bohr's inequality, Hadamard's Real Part Theorem.

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