Andriy A. Kondratyuk

A Carleman-Nevanlinna Theorem and Summation of the Riemann Zeta-Function Logarithm

CMFT 4 No.2 (2004), 391–403. [ISSN 1617-9447]

Abstract. A Carleman-Nevanlinna Theorem for a rectangle is proved. The theorem is applied to the summation of $\log|\zeta(s)|$ on the critical and other vertical lines, where $\zeta(s)$ is the Riemann zeta-function. In particular, let

$$I(\varepsilon) = \int_0^\infty e^{-\varepsilon t} \log\left|\zeta\left(\frac{1}{2}+it\right)\right| \, dt, \qquad \varepsilon > 0,$$

and let $\{\rho_j\}$ be non-trivial zeros of $\zeta(s)$, then

$$\frac{\pi}{2}\sum_j \left|\operatorname{Re}\rho_j - \frac{1}{2}\right| = I(+0) + \frac{\pi}{2},$$

where $I(+0) := \lim_{\varepsilon\to 0} I(\varepsilon)$. Thus, the Riemann hypothesis for $\zeta(s)$ holds if and only if $I(+0) = -\pi/2$.

Keywords. Meromorphic function, Carleman-Nevanlinna Theorem, summation, Riemann zeta-function, Euler product, Riemann hypothesis, Dirichlet series, Selberg class, L-function.

2000 MSC. Primary 30Dxx, 11M06; Secondary 11M41.

Received. February 5, 2004, in revised form September 30, 2004.