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Majorization of the Critical Points of a Polynomial by Its Zeros

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Abstract. Let $z_1, \ldots, z_n$ be the zeros of a polynomial $f(z)$ and let $\zeta_1, \ldots, \zeta_n$ be those of $zf'(z)$. Suppose that for both polynomials the zeros are labelled in order of non-increasing modulus. We show that

$$\sum_{\nu=1}^{k} |\zeta_\nu| \le \sum_{\nu=1}^{k} |z_\nu|, \qquad k = 1, \ldots, n,$$

which means that the moduli of the zeros of $f(z)$ weakly majorize those of $zf'(z)$. This refines the Gauss-Lucas Theorem. Moreover, this weak majorization is preserved if we replace $|\zeta_\nu|$ by $\psi(|\zeta_\nu|)$ and $|z_\nu|$ by $\psi(|z_\nu|)$ for $\nu = 1, \ldots, n$, where $\psi \circ \exp$ is any non-decreasing convex function on $\mathbb{R}$. Actually, we establish more general results which hold for a polynomial f and a certain multiplicative composition which may be interpreted as a Hadamard product of f with a polynomial from a certain class.

Keywords. Majorization, zeros of polynomials, critical points, Gauss-Lucas Theorem, inequalities.

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