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Zeros of Solutions of a Functional Equation

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Abstract. We consider the zeros of transcendental entire solutions f of the functional equation

$$\sum_{j=0}^{m} a_j(z) f(c^j z) = Q(z),$$

where $c \in \mathbb{C}$, $0 < |c| < 1$, and Q and the a_j are polynomials. Under a suitable hypothesis concerning the associated Newton-Puiseux diagram we show that the zeros of f are asymptotic to certain geometric progressions. More precisely, with this hypothesis there exist positive integers M and N such that the zero set can written in the form $\{z_{n,\mu} : \mu \in \{1, 2, \ldots, M\}, n \in \mathbb{N}\}$ where for each μ in $\{1, 2, \ldots, M\}$ there exists A_μ in $\mathbb{C} \setminus \{0\}$ with $z_{n,\mu} \sim A_\mu \, c^{-Nn}$ as $n \to \infty$. The proof is achieved by showing that f behaves asymptotically like a product of θ-functions. The hypothesis on the Newton-Puiseux diagram is satisfied, e.g., if for each positive σ and each real τ the line $\{(x, y) \in \mathbb{R}^2 : y = \sigma x + \tau\}$ contains at most two points of the form $(j, \deg(a_j))$. In particular, this is the case if all a_j are linear, in which case the above conclusion follows with $M = 1$ which means that the zeros are asymptotic to only one geometric progression. The hypothesis on the Newton-Puiseux diagram is also satisfied if $m = 1$. If $m = 1$ and $Q \equiv 0$, however, we have a much simpler and more precise result. We illustrate our results by a number of examples. In particular, we show that if the hypothesis on the Newton-Puiseux diagram is not satisfied, then the zeros of the solutions need not be asymptotic to a finite number of geometric progessions.

Keywords. q-difference equation, q-series, theta function, Newton-Puiseux diagram.

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