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Mapping Properties of Hypergeometric Functions and Convolutions of Starlike or Convex Functions of Order α

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Abstract. We determine the order of convexity of hypergeometric functions $z \mapsto F(a, b, c, z)$ as well as the order of starlikeness of shifted hypergeometric functions $z \mapsto zF(a, b, c, z)$, for certain ranges of the real parameters a, b and c . As a consequence we obtain the sharp lower bound for the order of convexity of the convolution $(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n$ when $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is convex of order $\alpha \in [0, 1]$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is convex of order $\beta \in [0, 1]$, and likewise we obtain the sharp lower bound for the order of starlikeness of $f * g$ when f, g are starlike of order $\alpha, \beta \in [1/2, 1]$, respectively. Further we obtain convexity in the direction of the imaginary axis for hypergeometric functions and for three ratios of hypergeometric functions as well as for the corresponding shifted expressions.

In the proofs we use the continued fraction of Gauss, a theorem of Wall which yields a characterization of Hausdorff moment sequences by means of (continued) g -fractions, and results of Merkes, Wirths and Pólya. Finally we state a subordination problem.

Keywords. Hadamard product, hypergeometric function, order of convexity, order of starlikeness, convexity in direction of the imaginary axis, continued fraction of Gauss, g -fraction, Hausdorff moment sequence.

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