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Normality Properties of the Family $\{f(nz) : n \in \mathbb{N}\}$

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Abstract. A family $\mathcal{F}$ of meromorphic functions on a plane domain D is called quasi-normal on D if each sequence S of functions in $\mathcal{F}$ has a subsequence which converges locally χ-uniformly on $D \setminus E$, where $E = E(S)$ is a subset of D having no accumulation points in D. The notion of quasi-normality was introduced by Montel. It generalizes the concept of normal family, which corresponds to $E = \emptyset$.

Chi-Tai Chuang extended the notion of a quasi-normal family further in an inductive fashion. According to Chuang, a family $\mathcal{F}$ of meromorphic functions on a plane domain D is Q_m-normal ($m = 0, 1, 2, \dots$) if each sequence S of functions in $\mathcal{F}$ has a subsequence which converges locally χ-uniformly on the domain $D \setminus E$, where $E = E(S) \subset D$ satisfies $E_D^{(m)} = \emptyset$. m is said to be the degree of $\mathcal{F}$. (Here $E_D^{(m)}$ is the m-th derived set of E in D.) In particular, a Q_0-normal family is a normal family, and a Q_1-normal family is a quasi-normal family.

This paper is devoted to the study of non-normal families generated by a single function. Let f be a nonconstant meromorphic function on $\mathbb{C}$, and write $f_n(z) = f(nz)$. Then the family $\Pi(f) = \{f_n : n \in \mathbb{N}\}$ is not normal on the unit disk. We examine the degree of non-normality of this family. We show that if f is a rational function, then $\Pi(f)$ is quasi-normal of exact order 1. In the opposite direction, if f is not rational, and there exist $a, b \in \hat{\mathbb{C}}$ such that the equations $f(z) = a$ and $f(z) = b$ have only a finite number of solutions in $\mathbb{C}$, then $\Pi(f)$ fails to be Q_m-normal for any value of m.

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