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Hyperholomorphic Functions

CMFT 1 No.1 (2001), 179–192. [ISSN 1617-9447]

Abstract. Let $\mathbb{H}$ be the algebra of quaternions and $\mathcal{C}\ell_{0,3}$ the Clifford algebra generated by e_1, e_2, e_3, subject to the conditions ${e_1}^2 = {e_2}^2 = {e_3}^2 = -1$ and $e_i e_k = -e_k e_i$ $(i, k = 1, 2, 3 : i \neq k)$. Embedding $\mathbb{H}$ in $\mathcal{C}\ell_{0,3}$, any element in $\mathcal{C}\ell_{0,3}$ may be written in the form $q_0 + q_1 e_3$, where $q_0, q_1 \in \mathbb{H}$. We thus write $\mathbb{H}_2$ rather than $\mathcal{C}\ell_{0,3}$. Let Q denote the projection operator given by $Q(q_0 + q_1 e_3) = q_1$. We are interested in the modified Dirac operator in $\mathbb{R}^4$, defined by

$$Mf(x) = Df(x) + \frac{2(Qf)'(x)}{x_3},$$

where $x = \sum_{i=0}^{3} x_i e_i$, $D = \sum_{i=0}^{3} e_i \partial/\partial x_i$ and $'$ designates the main involution in $\mathbb{H}_2$. An infinitely differentiable function $f\colon \Omega \to \mathbb{H}_2$ is called hyperholomorphic in an open set $\Omega \subset \mathbb{R}^4$, if $Mf(x) = 0$, for all $x \in \Omega$ with $x_3 \neq 0$. The second author noticed in [18] that the power function $x \to x^m$ is hyperholomorphic. A function $f\colon \Omega \to \mathbb{H}_2$ is called hyperbolic harmonic if $M\overline{M}f = 0$, where

$$\overline{M}f(x) = \overline{D}f(x) - \frac{2(Qf)'(x)}{x_3}.$$

Note that a real-valued function f is hyperbolic harmonic if and only if it satisfies the Laplace-Beltrami equation $x_3 \triangle f - 2\partial f/\partial x_3 = 0$, associated with the hyperbolic metric on $\mathbb{R}^4_+$. We show that f is hyperholomorphic if and only if both f and xf are hyperbolic harmonic. Hyperholomorphic functions are holomorphic Cliffordian functions, a concept introduced by G. Laville and I. Ramadanoff in [17]. Our main result states that, locally, holomorphic Cliffordian functions can always be written in terms of four hyperholomorphic functions. At last we give a representation theorem.

Keywords. Hyperholomorphic functions, hyperbolic metrics, monogenic, hypermonogenic, quaternions, generalized function theory.

2000 MSC. Primary 30G35; Secondary 30A05, 30F45.

Received. November 6, 2001.